Higher categorical models of linear logic

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Chapter 1

Introduction

1.1 In the beginning

Before explaining the subject of this thesis, I want to tell the story of how it came to be. I first encountered linear logic in 2022 while a student at the MPRI master in Computer Science at université Paris-Cité, in the course of Paul-André Melliès and Thomas Ehrhard, where I also heard some ideas about differential linear logic. Paul-André and Thomas's course put a big emphasis on the *categorical semantics* of linear logic. The idea that linearity and differentiation could be described in categorical manner reminded me of something else: I had been studying algebraic topology and homotopy the year before, and had also heard about *Goodwillie's calculus of functors*, a deep analogy applying ideas from differential calculus to computations involving functors between ∞-categories, the idea being that one could "approximate" a functor using a Taylor expansion. The unavoidable question that came to me then was:

> "Does Goodwillie's calculus of functors fit into a model of differential linear logic, in
> a possibly higher-categorical sense ?"

I talked about this idea with Mathieu Anel, Eric Finster, and especially Paul-André Melliès, who told me in an email:

> "As I told you, I have long been fascinated by this hypothesis that Goodwillie's
> calculus could be linked with differential linear logic, and I am glad you were
> seduced by the idea. On the other hand, like Mathieu I think we have a long road
> ahead before we can establish this link."

From this perspective, the background goal of my thesis has been to build this road, so that the question of this connection could one day be approached formally. Since Goodwillie's calculus is inherently ∞-categorical in nature, this meant generalizing the theory of categorical semantics of linear logic to the setting of higher categories and homotopy theory. I first approached this in the setting of homotopy type theory, building upon the work of my advisors on polynomial functors [Fin+21], then directly in the theory of ∞-categories developed by Joyal and Lurie. We will only get back to Goodwillie's calculus in chapter 8 as a perspective for future developments.

Because of the various technical notions and frameworks we rely on in this thesis, I thought it best to do domain-specific introductions in separate "survey chapters" for:

- linear logic (chapter 3),

- homotopy type theory (chapter 4),

- and the theory of ∞-categories (chapter 6).

In the rest of this introduction, we give a general overview of some of the motivations.

1.2 Linear logic and its semantics

Linear logic is an extension of classical logic (and/or intuitionistic logic) introduced by Girard in his seminal paper "Linear logic" [Gir87], in which the classical implication

$$A \implies B$$

is decomposed as

$$!A \multimap B,$$

where "$\multimap$" denotes a *linear implication*, and "!" an *exponential modality*. In linear logic, formulas are thought of as *resources*. Intuitively, the proofs of $A \multimap B$ correspond to ways of constructing the resource B from the resource A, and in particular one cannot duplicate the resource A. The formula "$!A$" allows one to duplicate elements of A, so that one may read $!A$ as "as many of A as one like."

When considering categorical semantics of linear logic, one must give interpretations for both of these logical connectives. The prototypical categorical model of linear logic is given by the category Rel in which:

- objects are sets $X, Y, \dots$

- morphisms $X \to Y$ are the relations $R \subseteq X \times Y$.

The formula $A \multimap B$ is interpreted as the product of sets $A \times B$, while the exponential modality is interpreted by taking multisets:

$$!A := \mathrm{Mul}\,(A) = \bigsqcup_{n \in \mathbb{N}} X^n / \mathfrak{S}_n.$$

The morphisms from $\mathrm{Mul}\,(A)$ to B in Rel (i.e. the relations $R \subseteq \mathrm{Mul}\,(A) \times B$) are called the *non-linear morphisms* of the model. One can build a category Rel $_{\mathrm{Mul}}$ in which:

- the objects are the sets,

- the morphisms $X \to Y$ are the relations $R \subseteq \mathrm{Mul}\,(X) \times Y$, i.e. the non-linear morphisms.

This category Rel $_{\mathrm{Mul}}$ is *cartesian closed*, and as such it is a model of the λ-calculus/of intuitionistic logic. Formally, Rel $_{\mathrm{Mul}}$ is the (co)Kleisli category for the comonad Mul on Rel. More generally, from any categorical model of linear logic one can recover a model of intuitionistic logic in this way, and from the point of view of categorical semantics it is in that sense that linear logic extends intuitionistic logic.

1.3 Normal functors and bicategorical models

Girard has noted that the first appearance of the decomposition

$$A \implies B \quad := \quad !A \multimap B$$

dates back to his article "Normal functors, power series and λ-calculus" [Gir88] in which he described a quantitative model of intuitionistic logic using *normal functors*. Given sets X and Y, a *normal functor* from Set^X to Set^Y is a functor $F : \mathrm{Set}^X \to \mathrm{Set}^Y$ that can be obtained using small coproducts and finite products of the projections. For instance:

$$\mathrm{Set}^1 \to \mathrm{Set}^1 \qquad\qquad \mathrm{Set}^3 \to \mathrm{Set}^2$$
$$X \mapsto \bigsqcup_{n\in\mathbb{N}} X^n \qquad\qquad (X,Y,Z) \mapsto (X \times Y \sqcup Z, X \sqcup Y^2 \times Z \sqcup 1)$$

Nowadays, normal functors are better known under the name of *finitary polynomial functors* (the non-finitary polynomials also allowing infinite products). Writing Norm for the category in which:

- the objects are sets,

- the morphisms $X \to Y$ are the natural isomorphism classes of normal functors from Set^X to Set^Y,

it can be shown that Norm is cartesian closed, with cartesian product of X and Y given by the disjoint union $X \sqcup Y$, and the internal hom from X to Y given by $\mathrm{Mul}\,(X) \times Y$: the same as in the category $\mathrm{Rel}_{\mathrm{Mul}}$! In other words, there is a natural bijection:

$$\mathrm{Norm}\,(X \sqcup Y, Z) \simeq \mathrm{Norm}\,(X, \mathrm{Mul}\,(Y) \times Z). \tag{1.1}$$

But since the morphisms in Norm are functors, it is natural to consider Norm not simply as a category but as a bicategory. Write NORM for the bicategory in which:

- the objects are sets,

- the morphisms $X \to Y$ are the normal functors from Set^X to Set^Y,

- the 2-morphisms $F \Rightarrow G$ are the (cartesian) natural transformations between functors.

Then NORM is surprisingly not cartesian closed in the bicategorical sense! This was already noticed by Girard [Gir88, Rmk 2.19]: consider for instance the case $X = \varnothing$, $Y = Z = 1$ and the functor

$$\mathrm{Set}^{\varnothing \sqcup 1} \simeq \mathrm{Set}^1 \to \mathrm{Set}^1$$
$$X \mapsto X^2$$

Under the bijection (1.1), this functor corresponds to the functor

$$F : \mathrm{Set}^{\varnothing} \simeq 1 \to \mathrm{Set}^{\mathbb{N}} \simeq \mathrm{Set}^{\mathrm{Mul}\,(1) \times 1}$$
$$* \mapsto \left(n \mapsto \begin{cases} 1 \text{ if } n = 2 \\ \varnothing \text{ otherwise} \end{cases}\right)$$

And while the functor $X \mapsto X^2$ has two automorphisms (the identity and the one given by $(x,y) \mapsto (y,x)$), the functor F only has one automorphism. In particular, the bijection (1.1) cannot be promoted to an equivalence of categories:

$$\mathrm{NORM}\,(X \sqcup Y, Z) \not\simeq \mathrm{NORM}\,(X, \mathrm{Mul}\,(Y) \times Z).$$

To solve this issue, my advisors Samuel Mimram and Thomas Seiller, together with Eric Finster and Maxime Lucas, replaced the set

$$\mathrm{Mul}\,(X) := \bigsqcup_{n \in \mathbb{N}} X^n / \mathfrak{S}_n$$

of multisets on X by a *groupoid*

$$\mathrm{HMul}\,(X) := \bigsqcup_{n \in \mathbb{N}} X^n /\!/ \mathfrak{S}_n$$

of *homotopy multisets* on X [Fin+21]. The notation $X^n /\!/ \mathfrak{S}_n$ denotes a *homotopy quotient*: While the traditional quotient simply identifies the lists (x, x, y) and (x, y, x), the *homotopy quotient* adds an *isomorphism* between them for every permutation that maps (x, x, y) to (x, y, x). For instance, while in $\mathrm{Mul}\,(X)$ we have $(x, x, y) = (x, y, x) = (y, x, x)$, in $\mathrm{HMul}\,(X)$ we have the following diagram of isomorphisms:
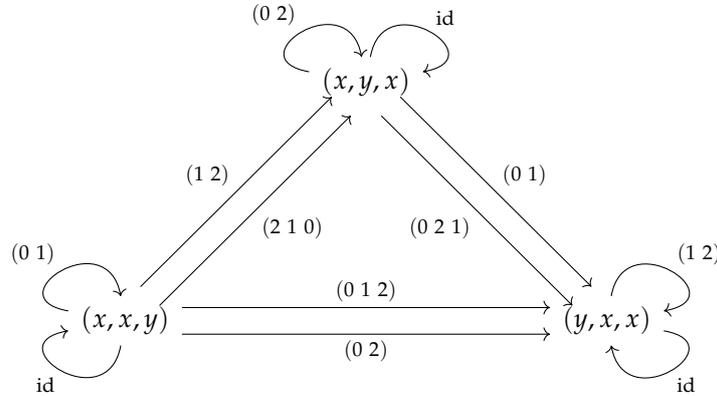
In order to do that efficiently, they work in the setting of homotopy type theory, where groupoids are a primitive notion, rather than one defined using sets.

The idea to replace ordinary multisets by homotopy multisets had been considered before in models of linear logic, under a different name: indeed $\mathrm{HMul}\,(X)$ is better known as $\mathrm{Sym}\,(X)$, the *free symmetric monoidal category* (or groupoid) on X.

1. For instance in 2007, Marcelo Fiore, Nicola Gambino, Martin Hyland and Glynn Winskel solved the issue in Girard's model by considering another generalization of normal functors that they called *generalized species of structure* [Fio+07]. While normal functors only make sense between categories of the form Set^X with X a set, generalized species can be defined between arbitrary presheaf categories $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. In their setting, the bicategory Esp of generalized species is constructed as a bicategory of non-linear morphisms for an exponential modality on the bicategory Prof of categories and profunctors (this is further explored by Fiore, Gambino and Hyland in their 2024 paper [FGH24], in which they also investigate the differential structure of this model). The pseudocomonad in question is precisely the extension of the pseudomonad $\mathrm{Sym}\,(-)$ on Cat to Prof (we will say more about generalized species in section 3.11.1).

2. In 2019, motivated by his study of game-theoretic models of linear logic, Paul-André Melliès introduced bicategories of *template games*, the simplest example of which being the

bicategory of groupoids and spans of isofibrations between them [Mel19]. Again in his model, the exponential comonad is an extension of Sym $(-)$ from Cat to template games.

These are two instances where a homotopy-theoretic concept, the Sym $(-)$ construction, appeared naturally in the study of linear logic. In Melliès's model, the condition of being an isofibration (a very important notion in homotopy theory) is interpreted in his setting as the strategies being "aware" of the symmetries existing in the games, giving yet another such instance.

Since then, the study of bicategorical models of linear logic has been expanded upon by various authors [Oli20; Gal21; CF23; COP24].

1.4   Linear logic in homotopty type theory

The starting goal of my PhD was to extend the homotopical model of intuitionistic logic described in [Fin+21] to a homotopical model of linear logic. Working in homotopy type theory, the fundamental objects of the theory are *types* rather than *sets*. Types contain not only elements, but also *paths* or *isomorphisms* between these elements, paths between paths (called *homotopies*), and so on. In particular, groupoids are nothing more than types in which there are no non-trivial homotopies between paths.

In homotopy type theory, the usual notion of category is too strict to represent the objects we are interested in: there should be a *type* of morphisms between not a *set* of morphisms. The composition law

$$\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \to \text{Hom}_{\mathcal{C}}(x, z)$$

is thus represented by a function between types, and we only require it to be associative up to isomorphism, or in other worlds we only ask for the data of a path

$$(h \circ g) \circ f \sim h \circ (g \circ f).$$

The reader familiar with the theory of bicategories may already expect what comes next: since composition is only associative up to isomorphism, the isomorphisms in question should also be required to satisfy some conditions of their own, called *coherence conditions*. These conditions in turn also correspond to the data of homotopies, themselves needing to be subject to further conditions...

Unfortunately, the problem of defining all the required coherences inside of homotopy type theory is currently open. Instead, we work with "incoherent" categories, only requiring low-dimensional coherences in our definitions. The resulting structure is called a *wild category*, a concept introduced in [CK18], which we further develop in sections 5.1 to 5.4. We then adapt Seely's axiomatization of categorical models of linear logic [See89] in section 5.5, and show that we can recover an actual 1-categorical model from any *wild model* in our sense.

After developing this theory, we construct a wild category of spans of types, and show that it is a wild model of linear logic theorem 5.7.33. We then go on to prove that the non-linear wild category associated to this model is equivalent to the cartesian closed bicategory of finitary polynomial functors defined in [Fin+21]. More precisely, we construct a family of models for the exponential modality "!", corresponding to different ways to allow the duplication of elements in a type. We do this by generalizing the construction of homotopy multisets to arities in a given *universe* $\mathcal{V}$, and show that the induced non-linear morphisms correspond to polynomial functors that are "$\mathcal{V}$-ary", i.e. functors constructed from small coproducts and $\mathcal{V}$-small products. In particular when $\mathcal{V}$ corresponds to finite sets, we recover the finitary polynomial functors.

Finally, we investigate under which condition on $\mathcal{V}$ our model can be promoted to a model of differential linear logic, building on the framework of bicategorical models of differential linear

logic of [FGH24]. We exhibit a necessary and sufficient condition in theorem 5.9.10, showing in particular the necessity of the strong condition that $\mathcal{V}$ must consist only of *discrete* types.

1.5 Linear logic in ∞-categories

Although the setting of wild categories was enough to describe an interesting homotopical semantics of linear logic, the fact that we needed to discard higher coherences was a bit unsatisfying. For instance, in linear logic the property that the resource $!A$ can be duplicated corresponds in categorical semantics to a commutative comonoid structure on $!A$:

$$!A \multimap !A \otimes !A.$$

In many situations, we can use this to *define* $!A$ to be the "cofree commutative comonoid on A", in which case we say that the exponential comonad is free. The impossibility of defining homotopy-coherent structures like commutative (co)monoids in homotopy type theory meant there was no way to define *free exponential modalities* in that setting. In particular, comonad $\mathrm{HMul}\,(-)$ should be free in that sense, but there is no way to even *state* this in homotopy type theory.

Because of this limitation, the next logical step was to work directly at the level of ∞-categories. Even though the technical requirements were heavy, nowadays the theory has been developed thoroughly enough, in great part in Lurie's books [Lur09; Lur17], that one can work very abstractly as if doing ordinary category theory, and seldom going back to low-level definitions. Leveraging this body of work, we managed to define ∞-categorical models of linear logic.

The first question of course was: "what should be an ∞-categorical model of linear logic ?" In the 1-categorical setting, many axiomatizations exist for the categorical structures required to model linear logic (we review those in section 3.4). The easiest one to transport to the ∞-categorical setting was that of *linear/non-linear adjunction*. The principle behind this axiomatization is to package in the very definition of categorical model both the *linear category* (with morphisms $A \multimap B$) and the *non-linear category* (with morphisms $!A \multimap B$), by means of a monoidal adjunction:

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$$

between a cartesian category $\mathcal{M}$ and a symmetric monoidal category $\mathcal{L}$.

Because this definition is very abstract, and relies only on concepts that already have well-understood ∞-categorical counterparts (adjunctions, monoidal structures...), we could essentially phrase it in the same way to *define* the notion of ∞-categorical model of linear logic (definition 7.1.7). Similarly, we could define free exponential comonads by adapting the 1-categorical definition (definition 7.1.11), and showed than any such free exponential induces an ∞-categorical linear/non-linear adjunction (proposition 7.1.12).

In that setting, we managed to give two criteria for the existence of such free exponential comonads: first for ∞-categories with enough limits (proposition 7.1.13), and then for ∞-categories with enough colimits, under an additional assumption of presentability theorem 7.1.15. Then, building upon Lurie's work on the tensor product of ∞-categories with colimits, we managed to apply these results to define ∞-categorical counterparts to the model of generalized species of [FGH24] and its variants described in Galal's thesis [Gal21]. We also noted how to link these models with the wild categorical models of chapter 5, describing perspectives for future work there (remark 7.2.31 and the discussion that follows).

Finally, we turned to ∞-categorical counterparts of linear algebraic models. The ∞-categorical generalizations of abelian groups are called *spectra*, and they have been studied for several decades in algebraic topology, in particular because they are known to represent cohomology theories (a fact known as Brown's representability theorem). Similarly, there is also a generalization of rings and modules, known as ring spectra and module spectra, and it turns out that module spectra over an ordinary ring R correspond precisely to chain complexes of ordinary R-modules. In section 7.3, we build ∞-categorical models of linear logic in spectra and module spectra, generalizing the well-known 1-categorical models in abelian groups, vector spaces and modules.

1.6 Structure of the thesis

As explained above, in this thesis we will explore categorical models of linear logic both in the setting of homotopy type theory and in the setting of ∞-categories. We tried our best to make this manuscript self-contained, assuming only a background in category theory. To that end, we provide extensive surveys of the various concepts we encounter.

Beginning in chapter 2, we give an overview of various notions from higher category theory and homotopy theory, notably explaining the philosophical and technical links between these two theories. At the end of the chapter, we also survey the indexed/fibred correspondence: an ubiquitous principle that will appear repeatedly throughout this thesis under various guises.

In chapter 3, we survey the syntax and categorical semantics of linear logic. We give a novel way to present the relational model of linear logic in section 3.6, before presenting other known and less well-known models in Scott domains, vector spaces, and finally generalized species, closing with a discussion of differential linear logic. This chapter also introduces the correspondence between intensional and extensional points of view on relations and their generalizations, elaborating on a terminology due to [FGP23].

The next two chapters are concerned with the categorical semantics of linear logic *inside* of homotopy type theory. Chapter 4 is a survey of type theory and homotopy type theory, first defining the fundamental concepts of dependent type theory, then building up to the univalence axiom and the homotopical interpretation of types. This leads into chapter 5, where we elaborate the theory of wild categories first introduced in [CK18]: a way to go around the limitations of homotopy type theory as regards to homotopy-coherent structures. Equipped with this homotopical category theory, we describe a wild-categorical of linear logic based on spans of types, and show that its non-linear morphisms correspond to the polynomial functors in types defined in [Fin+21]: this is essentially the content of our first article [HM24]. We then expand upon this content to show under which conditions this model can be extended to a wild-bicategorical model of differential linear logic, building on the work of [FGH24].

The two subsequent chapters finally approach the theory of ∞-categories. In Chapter 6, we survey Joyal and Lurie's theory, providing explicit references to Lurie's works [Lur09; Lur17; Lur18] and some others for every result, and also prove some folklore results about internal homs. This allows us in chapter 7 to define ∞-categorical models of linear logic by analogy with the axiomatizations presented in chapter 3. We then go on to construct ∞-categorical variants to all the models described in chapter 3, and comment on some connections with the homotopy type theoretical models described in chapter 5: this contains essentially the contents of our second article [HM25], together with some different proofs and new results.

To conclude, we explore some perspectives for future research in chapter 8, finally saying a bit more about Goodwillie's calculus of functors.

1.7   Notations used throughout the text

Most of the notations in this thesis will be introduced as they are needed, but we include here a list of notations that we will use throughout in various different contexts due to their ubiquity. For readers reading the pdf version of this text, note that most notations and technical terms are clickable, and will link back to their definition when clicked, thanks to the use of the *Knowledge* LATEXpackage. Because of this, we advise the reader to use a pdf-reading application that displays previews of hyperlinks for ease of navigation.

- given a category $\mathcal{C}$, the Yoneda embedding

$$\mathcal{C} \to \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}\,)$$
$$x \mapsto \mathrm{Hom}_{\mathcal{C}}(-, x)$$

  will be denoted with the japanese character よ (read "yo").

- Given two things x and y (sets, objects in a category, types in type theory), their cartesian product is denoted $x \times y$ or $x \,\&\, y$

- Given maps/morphisms/arrows $f : x \to y, g : x \to z$, the induced map into the cartesian product will be denoted
$$\langle f, g \rangle : x \to y \times z.$$

- In the case of the nullary product, we write $\langle \rangle : x \to 1$ for the unique map to the terminal object.

- We write $\Delta_x : x \to x \times x$ for the diagonal map $\Delta_x := \langle \mathrm{id}_x, \mathrm{id}_x \rangle$, and more generally
$$\Delta_x : x \to x^y$$

  for the diagonal map to the internal hom in cartesian closed situations.

- Given two things x and y (sets, objects in a category, types in type theory), their coproduct is denoted $x \sqcup y$

- Given maps $f : x \to z, g : y \to z$, induced map from the coproduct is denoted
$$[f, g] : x \sqcup y.$$

- In the case of the nullary coproduct, we write $[] : \varnothing \to x$ for the unique map from the initial object.

- We write $\delta_x^y := \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$ for the Kronecker delta.

- We write Set for the category of sets and functions.

- We write Cat for the category of categories and functors.

- We write $\mathrm{Bool} := \{0, 1\}$ for the set of booleans, 0 representing "false" and 1 representing "true"

- Given a set X, we write $P(X) := \{U \subseteq X\}$ for the *powerset* of X.

- If E is a poset, we write $P(E) := \{U \subseteq E \mid U \text{ is downward-closed}\}$, so as to be compatible with the definition above when E is a discrete poset.

- When X is a set carrying some structure (e.g. a monoid, a group, a topological space), we write $|X|$ for the underlying set, forgetting the structure. More generally we write $|-|$ for forgetful functors.

- When $\mathcal{C}$ is a category and x, y are objects of $\mathcal{C}$, $\text{Hom}_{\mathcal{C}}(x, y)$ or $\mathcal{C}(x, y)$ denotes the set of morphisms from x to y in $\mathcal{C}$.

- Given functions or morphisms $f : x \to y$, $g : y \to z$, their composition is denoted either $g \circ f$ or $f ; g$.

Chapter 2

Philosophico-homotopical preliminaries

2.1 What is meant by "higher categories"

Categories organize mathematical objects by specifying morphisms between them and how they compose. The idea of higher categories is that, to consider morphisms themselves as interesting objects of study, one may need to add "morphisms between morphisms", and these new "2-morphisms" might themselves have "3-morphisms" between them, and so on. To understand what this might look like, it is perhaps best to look at an example: category theory itself. Since categories themselves are mathematical objects, they form a category, whose morphisms are functors. However this "category of categories" is unsatisfying in some aspects (even pushing aside size issues).

Indeed, fixing categories $\mathcal{C}$ and $\mathcal{D}$, functors from $\mathcal{C}$ to $\mathcal{D}$ themselves form a category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, whose morphisms are natural transformations. One of the main lessons of category theory is the following:

Slogan 2.1.1 (*Principle of equivalence*)**.** When comparing objects of a category, the right notion of "sameness" is given by isomorphism rather than by equality. Isomorphism is the only *categorically meaningful* notion of sameness.

So two functors should be considered "equal" whenever they are isomorphic. In particular, a functor $F : \mathcal{C} \to \mathcal{D}$ should be considered an "isomorphism of categories" when there is some $G : \mathcal{D} \to \mathcal{C}$ such that

- $F \circ G$ is naturally isomorphic to $\operatorname{id}_{\mathcal{D}}$,

- $G \circ F$ is naturally isomorphic to $\operatorname{id}_{\mathcal{C}}$.

In practice, such a functor F is called an *equivalence of categories*, and we keep the term "isomorphism of categories" for the stricter notion where we ask $F \circ G = \operatorname{id}_{\mathcal{D}}$ and $G \circ F = \operatorname{id}_{\mathcal{C}}$. However when practicing category theory, it quickly becomes clear that the right way to compare categories is through equivalence rather than by isomorphism (so much so that during a time, some people in the category theory community started using the adjective "evil" to qualify constructions that did not respect the principle of equivalence – see the discussion at [nLa26]).

A way to solve this issue is to say that there is not really such a thing as a "category of categories", but rather a "2-category of categories".

Definition 2.1.2. A 2-category consists of

1. a class of objects Ob,

2. for every pair of objects (x, y), a category $\mathrm{Hom}(x, y)$ of morphisms,

3. for every $x \in \mathrm{Ob}$, a distinguished morphisms $\mathrm{id}_x \in \mathrm{Hom}(\mathrm{x},\mathrm{x})$

4. for every $x, y, z \in \mathrm{Ob}$, a composition functor

$$- \circ - : \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z),$$

5. such that composition is associative and unital.

The morphisms between $f, g \in \mathrm{Hom}(x, y)$ in the category $\mathrm{Hom}(x, y)$ are called "2-morphisms".

By definition, a 2-category consists of objects, morphisms between objects, and so-called 2-morphisms between morphisms, together with different composition laws and coherence axioms. In our case, the objects are categories, the morphisms are functors, and the 2-morphisms are the natural transformations.

In a 2-category, the only meaningful notion of "sameness" between morphisms is that of 2-isomorphism, so the canonical notion of isomorphism is that of a morphism invertible up to a 2-isomorphism, which in the case of the 2-category of categories yields the usual notion of equivalence of categories.

There is still one issue with the definition of 2-category: the operation of composition is required to be "associative and unital". For instance for associativity, this means an equality of functors

$$(- \circ -) \circ - = - \circ (- \circ -).$$

But the whole point of introducing 2-categories was to compare functors only through natural isomorphism. To solve this issue, the definition of 2-category can be relaxed to only asking for a natural isomorphism

$$\alpha : (- \circ -) \circ - \simeq - \circ (- \circ -)$$

and similar natural isomorphisms witnessing "unitality up to isomorphism". However this new data may add some unwanted behaviour: for instance, given 4 composable morphisms

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \xrightarrow{\ h\ } t \xrightarrow{\ k\ } u$$

there are multiple ways to witness the isomorphism

$$((k \circ h) \circ g) \circ f \simeq k \circ (h \circ (g \circ f))$$

as shown in the pentagon below (called *Mac Lane's pentagon*):

So we also need to ask for all possible ways to compose the α isomorphisms to coincide. It turns out to only be sufficient to ask for the above pentagon to commute. Adding similar conditions for unitality isomorphisms, we get to the notion of *bicategory*. Bicategories are the "right way" to do 2-categories: every 2-category can be seen as a bicategory, and the definition of bicategory respects the principle of equivalence.

Furthermore, every category can be seen as a 2-category whose Hom-categories are "discrete" (i.e. only have identity 2-morphisms), and in particular as a bicategory. In particular, the previously mentioned "category of categories" can be seen as a bicategory, and unsurprisingly it is not equivalent (in an appropriate bicategorical sense) to the bicategory of categories, functors and natural transformations: despite having the same objects (categories), the notion of isomorphism between objects is not. More formally:

Fact 2.1.3. *The canonical* (*bi*)*functor from the bicategory of categories, functors and equalities to the bicategory of categories, functors and natural transformations is not an equivalence of bicategories. It maps equivalences of categories to isomorphisms* (*in the appropriate bicategorical sense of isomorphism*).

From this point of view, rather than considering as "evil" constructions that do not respect the principle of equivalence, one should think of them as constructions on the bicategory of categories, functors and equalities of functors. Going even further, one could argue that the only actually "evil" thing in this situation is to use the name "category" for the objects of two bicategories that are not equivalent. Following this idea, people working in homotopy type theory have coined the terms *precategory* and *strict categories* to denote categories considered up to isomorphism [Uni13, Chapter 10].

Finally, a nice coherence theorem states that in most situations we can simply work with 2-categories anyway.

Fact 2.1.4. *Every bicategory is equivalent* (*in a bicategorical sense*) *to a 2-category.*

While bicategories provide a satisfying setting to talk about categories, they suffer the same issue that categories suffered in the same place: the "bicategory of bicategories" isn't "right" in the same way that the "category of categories" wasn't. When considering bicategories as objects of study, one really needs to start talking about 3-categories or "tricategories": categories with objects, morphisms, 2-morphisms, and 3-morphisms between 3-morphisms. But then we lose nice coherence results: not every general "tricategory" is equivalent to a 3-category as not every coherence isomorphism can be strictified to an equality like they could be for bicategories. Moreover, the combinatorics of higher cells quickly becomes unmanageable when going to higher dimensions (the very definition of tetracategory taking not less than 50 pages [Tri06]).

The solution to this conundrum was finally found, not through a piling up of yet more complex definitions, but through a beautiful connection with what was then a subbranch of the mathematical field of topology: **homotopy theory**.

2.2   Ideas from homotopy theory

The goal of this section is to explain to core concepts behind homotopy theory, their motivations, and how this all links back to (higher) category theory.

Homotopy theory has its roots in algebraic topology: a field of mathematics aiming to study topological spaces through the use of algebraic invariants. For instance, the "number of holes" of a topological space does not depend on its homeomorphism class: two homeomorphic topological spaces have the same "number of holes" since, intuitively, a continuous map cannot "tear apart" the space to increase or decrease said number. Hence the "number of holes" of a topological

space is a *numerical invariant*: a number invariant under some equivalence relation (here, that of being homeomorphic).

Rather than being interested in merely numerical invariants, algebraic topology studies algebraic invariants: for instance, to every topological space X can be associated a sequence of groups $(H_n(X))_{n\in\mathbb{N}}$ called the homology groups of X. Given a basepoint $x \in X$, one can also build *homotopy groups* $(\pi_n(X,x))_{n\in\mathbb{N}}$. The groups $H_n(X)$ and $\pi_n(X,x)$ both detect "n-dimensional holes" in X, each in a slightly different interpretation.

Remark 2.2.1. The study of homology groups was one of the motivations for the introduction of category theory by Saunders Mac Lane and Samuel Eilenberg in 1945 [EM45].

The operation $X \mapsto H_n(X)$ is functorial in X: the n-th homology group is a functor from the category of topological space to the category of (abelian) groups. Similarly, $\pi_n(-)$ is a functor from the category of pointed topological spaces to groups. In particular, they map homeomorphic topological spaces to isomorphic groups: it is in that sense that homology and homotopy groups are algebraic invariants. However these invariants are far from being complete, for instance the euclidean spaces $\mathbb{R}^k, k \in \mathbb{N}$ have the same homology and homotopy groups despite being pairwise non-homeomorphic as topological spaces. In fact those groups are invariants for a way weaker equivalence relation.

Definition 2.2.2. Let $f, g : X \to Y$ be continuous maps between topological spaces. A *homotopy* from f to g is a continuous map

$$H : [0,1] \times X \to Y$$

such that $H(0, -) = f$ and $H(1, -) = g$. The maps f and g are said to be *homotopic* if there exists a homotopy between them, in which case we write $f \sim g$.

Definition 2.2.3. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there exists $g : Y \to X$ such that $f \circ g \sim \text{id}_Y$ and $g \circ f \sim \text{id}_X$. In that case we say X and Y are homotopy equivalent and we write $f : X \simeq Y$ or simply $X \simeq Y$.

Remark 2.2.4. The reader may already notice the similarity between homotopies in topology and natural isomorphisms in category theory, and between homotopy equivalences and equivalences of categories.

Fact 2.2.5. *If $f \sim g$, then $\pi_n(f) = \pi_n(g)$ and $H_n(f) = H_n(g)$.*

Corollary 2.2.6. *If $f : X \simeq Y$, $\pi_n(X,x) \simeq \pi_n(Y, f(x))$ (for all $x \in X$) and $H_n(X) \simeq H_n(Y)$.*

For instance, the space $\mathbb{R}^k$ is homotopy equivalent to a singleton space because the identity map of $\mathbb{R}^k$ can be "continuously deformed" to yield a constant map: we say that the space $\mathbb{R}^k$ is *contractible*.

The notion of homotopy is intimately linked with the very definition of *homotopy groups*: the group $\pi_1(X,x)$ (also called the fundamental group) is defined as the set of maps $\gamma : [0,1] \to X$ such that $\gamma(0) = x = \gamma(1)$ (i.e. *loops* from x to x in X), quotiented by the relation $\gamma_1 \sim \gamma_2$ if there is a homotopy H from γ_1 to γ_2 that preserves the endpoints, i.e.

$$H(0,t) = \gamma_1(t)$$
$$H(1,t) = \gamma_2(t)$$
$$H(t,0) = x$$
$$H(t,1) = x$$

The group structure comes from concatenation and reversal of paths. More generally, the group $\pi_n(X,x)$ is defined by looking at equivalence classes of maps $[0,1]^n \to X$ that map their boundary
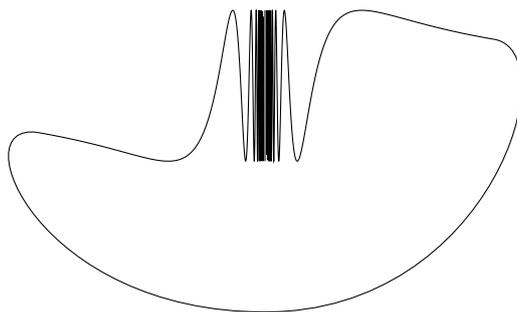
Figure 2.1: The Warsaw circle

to x, or equivalently maps $S^n \to X$ where S^n is the n-dimensional sphere (though the group structure is harder to see in this latter definition).

Example 2.2.7 (Homotopy groups of $\mathbb{R}^k$)**.** Since $\mathbb{R}^k$ is contractible, its homotopy groups are the same as the homotopy groups of the singleton space. As there is a unique map from any n-sphere into the singleton space, $\pi_n(\mathbb{R}^k, x)$ is isomorphic to the trivial group for every $x \in \mathbb{R}^k$.

Example 2.2.8 (Fundamental group of the circle)**.** Consider the circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Its fundamental group is $\pi_1(S^1, (1, 0)) \simeq \mathbb{Z}$. Intuitively, a path $\gamma \in \pi_1(S^1, (1, 0))$ is determined by the number of times it "loops around" the circle, together with its direction (clockwise or counterclockwise), hence by an integer (signed number of turns).

The fact that $\pi_1(S^1) \not\simeq \pi_1(\mathbb{R}^k)$ detects the fact that the circle contains a "1-dimensional hole" while $\mathbb{R}^k$ does not.

Slogan 2.2.9. In general, the number of generators of $\pi_n(X, x)$ count the number of "n-dimensional holes" in the path-connected component of x in X.

Because homotopy groups are closely linked to the very notion of homotopy of maps, it is no surprise that they are an invariant with respect to homotopy equivalence. However since the homotopy groups of X only detect "maps from spheres into X", they cannot distinguish more pathological spaces.

Example 2.2.10 (The Warsaw circle)**.** The Warsaw circle W (fig. 2.1) is a topological space built as a subspace of $\mathbb{R}^2$ obtained as a union of:

- the graph of the map $x \mapsto \sin(\frac{1}{x})$ around the origin,

- the vertical line $\{0\} \times [-1, 1]$ joining the two connected components of the previous graph,

- a curve joining the endpoints of the previous graph.

Despite looking like a circle, there is no map $\gamma : [0, 1] \to W$ that does a complete loop around it. This is because such a loop could not be continuous through the graph of the sine function at $x \approx 0$. As a consequence, $\pi_1(W, (0, 0))$ is a trivial group, and so are the other homotopy groups of W: the homotopy groups cannot distinguish W and a discrete space with two points (because W has two connected components), although they are not homotopy equivalent.

When the homotopy groups "believe" that a map is an isomorphism, that map is said to be a weak homotopy equivalence.

Definition 2.2.11. A continuous map $f : X \to Y$ between topological spaces is a *weak homotopy equivalence* if for every $x \in X$ and $n \in \mathbb{N}$, the induced homomorphism of groups

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism.

Even though homotopy groups are defined very differently than homology groups, homology turns out to be invariant under weak homotopy equivalence.

Fact 2.2.12 ([Hat02, Prop 4.21]). *If $f : X \to Y$ is a weak homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is a group isomorphism for every $n \in \mathbb{N}$.*

In that sense, the main objects of study of algebraic topology really are topological spaces *up to weak homotopy equivalence*.

Definition 2.2.13. The *homotopy type* of a topological space X is its equivalence class under weak homotopy equivalence.

The discrepancy between homotopy equivalence and weak homotopy equivalence motivated J. C. Whitehead in 1949 to introduce a class of topological spaces called CW-complexes, for which the notions actually coincide [Whi49]. A CW-complex is a space built inductively, by specifying first a set of vertices (0-cells), then edges connecting those vertices (1-cells), then disks glued along those edges (2-cells), balls glued along disks (3-cells), and more generally n-dimensional balls glued along lesser-dimensional ones.

Example 2.2.14. A discrete set of points is a 0-dimensional CW-complex. A graph determines a 1-dimensional CW-complex.

Fact 2.2.15 (*Whitehead's theorem*). *A continuous map $f : X \to Y$ between CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

Although the notion of CW-complex may for now seem arbitrary, it turns out that most non-pathological topological spaces encountered in mathematics are homotopy equivalent to CW-complexes (for instance, every manifold is even homeomorphic to a CW-complex). Moreover, we have the following:

Fact 2.2.16. *Every topological space is weakly homotopy equivalent to a CW-complex. In other words, every topological space has the homotopy type of a CW-complex.*

In that sense, everything that homotopy groups have to say about topology can be understood by looking only at CW-complexes. Given the very combinatorial nature of CW-complexes, this gives a first insight into the combinatorial — rather than topological — nature of homotopy theory.

Taking a step back from all this topology, the reader might notice a connection between the notion of a CW-complex and category theory. After all, a category is given by a set of vertices and edges, just like a CW-complex, and in a CW-complex, paths can be concatenated, just like morphisms in a category can be composed. And as mentioned in remark 2.2.4, homotopies and homotopy equivalences are very reminiscent of natural isomorphisms and equivalences of categories respectively.

More formally, to every topological space X, one can associate a category $\Pi(X)$ whose objects are the points of X and whose morphisms from x to y are the homotopy classes of paths from x to y in X. Identities are given by the constant paths, and composition is given by composition

of paths. Since paths are reversible, the morphisms in $\Pi(X)$ are all invertible, so that $\Pi(X)$ is actually a groupoid. Π defines a functor from the category of topological spaces and continuous maps to the category of groupoids and functors.

Going the other direction, one can associate to every category C a topological space $|C|$ called its *geometric realization*. $|C|$ is built as a CW-complex whose vertices are the points of C, whose edges are specified by the morphisms in C, and higher cells witness the commutation of diagrams in C. The space $|C|$ has the property that its homotopy groups $\pi_n(|C|, x)$ are trivial for $n > 1$ and every x.

Definition 2.2.17. A *homotopy n-type* is a homotopy type whose homotopy groups π_k vanish for $k > n$.

The formal link between groupoids and homotopy 1-types can be formulated as follows.

Fact 2.2.18. *The functors Π and $|-|$ induce an adjunction*

$$\mathrm{Cat} \; \underset{\Pi}{\overset{|-|}{\underset{\perp}{\rightleftarrows}}} \; \mathrm{Top}$$

between the category Cat *of categories and functors, and the category* Top *of topological spaces and continuous maps. This adjunction restricts to an adjunction*

$$\mathrm{Gpd} \; \underset{\Pi}{\overset{|-|}{\underset{\perp}{\rightleftarrows}}} \; \mathrm{Top}_{\leq 1}$$

between the category Gpd *of groupoids and the subcategory* $\mathrm{Top}_{\leq 1}$ *of* Top *on the spaces whose homotopy type is a 1-type.*

Moreover, for every $X \in \mathrm{Top}_{\leq 1}$, the counit map

$$|\Pi(X)| \to X$$

is a weak homotopy equivalence, and for every $G \in Gpd$, the unit map

$$G \to \Pi(|G|)$$

is an equivalence of groupoids.

The last part of this statement can be understood as saying that there is an equivalence of categories "up to natural weak equivalences" rather than "up to natural isomorphism".

Notice that this fact refers to the "category of groupoids" and not the "bicategory of groupoids", but still refers to the notion of equivalence of groupoids (whose definition relies on natural isomorphisms of functors). This result really says that groupoids and homotopy 1-types are one and the same, when they are considered up to their canonical equivalence relation.

The deeper insight here is that general homotopy types should correspond to higher-dimensional groupoids, and more precisely that homotopy n-types should correspond to n-groupoids. The point of a space correspond to objects in a higher groupoid, the paths correspond to isomorphisms, the homotopies correspond to 2-isomorphisms, and one can define homotopies between homotopies and so on to represent higher dimensional isomorphisms. This idea has become one of the core principles behind higher category theory, and a litmus test for definitions of higher categories.

Slogan 2.2.19 (The *homotopy hypothesis*)**.** Any suitable definition of n-category (for $n \in \mathbb{N} \cup \{\infty\}$) should satisfy that n-groupoids "correspond exactly" to homotopy n-types (where "homotopy ∞-type" means "arbitrary homotopy type").

A philosophical consequence of this principle is that higher groupoids may actually be way easier to define than more general higher categories.

2.3   Categorical homotopy theory

The categorical approach to the kind of situations we explored in the previous section is to consider general categories equipped with a notion of "weak equivalence".

Definition 2.3.1. A *relative category* is a category $\mathcal{C}$ equipped with a class $\mathcal{W}$ of morphisms in $\mathcal{C}$ called the *weak equivalences*, that contains all identity morphisms.

Given a relative category $(\mathcal{C}, \mathcal{W})$, what we are really interested in are not the morphisms in $\mathcal{C}$ but the morphisms in a modified version of $\mathcal{C}$ where the morphisms in $\mathcal{W}$ have been "formally inverted".

Definition 2.3.2. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. A *localization* of $\mathcal{C}$ at $\mathcal{W}$ is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a *localization functor* $Q : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ such that

1. for every morphism $w \in \mathcal{W}$, $Q(w)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$,

2. for every category $\mathcal{D}$, the precomposition functor induces an equivalence of categories

$$(- \circ Q) : \mathrm{Fun}\,(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \simeq \mathrm{Fun}\,_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}\,_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ on those functors that map every morphism $w \in \mathcal{W}$ to an isomorphism in $\mathcal{D}$.

Fact 2.3.3. *Localizations always exist: they can be constructed by taking the same objects of $\mathcal{C}$, and as morphisms zig-zags of morphisms in $\mathcal{C}$*

$$a \longrightarrow \bullet \xleftarrow{\in \mathcal{W}} \bullet \longrightarrow \bullet \dashrightarrow \bullet \xleftarrow{\in \mathcal{W}} \bullet \longrightarrow b$$

where all the left-pointing maps belong to $\mathcal{W}$, up to a well-chosen equivalence relation.

   Their definition by a universal property guarantees that localizations are unique up to equivalence of categories unique up to natural isomorphism.

The previous concrete construction of localizations using zig-zags of morphisms makes them extremely difficult to work with in practice. However in many situations of interest, additional structures on the relative category $(\mathcal{C}, \mathcal{W})$ can be used to find simpler descriptions of the localizations $\mathcal{C}[\mathcal{W}^{-1}]$. A general framework is given by Quillen's notion of *model categories* (for which a standard reference is Hirschhorn's book [Hir09]).

Definition 2.3.4. A *model structure* on a relative category $(\mathcal{C}, \mathcal{W})$ is the data of two other classes of morphisms Fib called "fibrations", and Cof called "cofibrations", that satisfy a lot of niceness assumptions.

   A *model category* is a complete and cocomplete relative category equipped with a model structure.

Example 2.3.5 (*Quillen model structure*)**.** There is a model structure on the category Top of topological spaces where the weak equivalences $\mathcal{W}$ are the weak homotopy equivalences.

Example 2.3.6 (*Canonical model structure*)**.** There is a model structure on the category Cat of categories such that the weak equivalences are the equivalences of categories.

The intuition behind fibrations and cofibrations in model categories can be a bit hard to explain in general, and we will see some examples in chapter 6. Importantly, these classes of maps can be used to define a notion of *homotopy between maps* in a any model category. Technically, two such notions can be defined ("left homotopies" and "right homotopies"), but they behave well when restricted to "good" objects:

Definition 2.3.7. Let $(\mathcal{C}, \mathcal{W}, \mathrm{Fib}, \mathrm{Cof})$ be a model category. An object $x \in \mathcal{C}$ is said to be

- *fibrant* if the unique map $x \to *$ to the terminal object is a fibration,

- *cofibrant* if the unique map $\varnothing \to x$ from the initial object is a cofibration,

- *bifibrant* if it is both fibrant and cofibrant.

Fact 2.3.8. *In a model category $\mathcal{C}$, let x be a cofibrant object and y a fibrant object. Then the relations "being left homotopic" and "being right homotopic" on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ coincide and they are equivalence relations, denoted $\sim$.*

Example 2.3.9. In the Quillen model structure on Top, every space is fibrant, and the cofibrant objects are the (retracts of) CW-complexes. The left and right homotopy relations coincide with the previously defined notion of homotopy between maps when the domain is a (retract of) a CW-complex.

Example 2.3.10. In the canonical model structure on Cat, every object is bifibrant. The homotopy relation coincides with "being naturally isomorphic".

Fibrant and cofibrant objects in a model category can be used to give a very nice description of the localization $\mathcal{C}[\mathcal{W}^{-1}]$.

Fact 2.3.11. *The relation of being homotopic is a congruence with respect to composition between bifibrant objects. In other words, given a diagram*

$$x \xrightarrow[f']{f} y \xrightarrow[g']{g} z$$

with x, y, z bifibrant and $f \sim f', g \sim g'$, we have

$$g \circ f \sim g' \circ f'.$$

Definition 2.3.12. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category whose objects are the bifibrant objects of $\mathcal{C}$, and where

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x, y) := \mathrm{Hom}_{\mathcal{C}}(x, y) / \sim .$$

Fact 2.3.13. *There is an equivalence of categories*

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathcal{C}[\mathcal{W}^{-1}].$$

In the case of topological spaces, this means that the "category of homotopy types" Top $[\mathcal{W}^{-1}]$ is equivalent to the category of (retracts of) CW-complexes and homotopy classes of continuous maps.

In the case of categories, this means that formally inverting equivalences of categories in Cat yields the same thing as considering the category of categories and isomorphism classes of functors.

These facts may come as a surprise: they seem to imply that the notion of *weak equivalence* "knows about" the natural notion of homotopy between morphisms. It is not obvious at all in category theory for instance, that the notion of natural isomorphism can be derived from the notion of equivalence of categories, as the latter is typically defined using the former. Here is a way to define the relation of "being naturally isomorphic" that relies only on equivalences of categories and equalities of functors:

Fact 2.3.14. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. F and G are naturally isomorphic if and only if there exists a factorization*

$$
\begin{array}{ccc}
 & \mathcal{C} & \\
{}^{[\mathrm{id}_\mathcal{C},\mathrm{id}_\mathcal{C}]}\nearrow & \simeq \Big\uparrow & \\
\mathcal{C} \sqcup \mathcal{C} \xrightarrow{\ i\ } & \mathcal{E} & \xrightarrow{\ p\ } \mathcal{D} \\
 & \underset{[F,G]}{} &
\end{array}
$$

where the functor i is injective on objects and the functor $\mathcal{E} \to \mathcal{D}$ is an equivalence of categories. Note that this diagram is required to commute strictly, and not just up to natural isomorphism.

The usual notion of natural isomorphism can be recovered by taking $\mathcal{E} := I \times \mathcal{C}$, where I is the category with two objects and a single isomorphism between them (the "walking isomorphism category"), and $\mathcal{C} \sqcup \mathcal{C} \to \mathcal{C} \times I$ given by the endpoint inclusions.

Actually, generalizing the theory of localizations to bicategories, it can be shown that localizing the bicategory of

- categories,

- functors,

- and identity 2-morphisms

at the class of equivalences of categories yields the bicategory of

- categories,

- functors,

- and natural isomorphisms.

So equivalences of categories can recover not only the notion of "being naturally isomorphic", but also the full structure of natural isomorphisms and the way they compose in the bicategory of bicategories. From the point of view of model categories, the difference between considering equality between functors or natural isomorphism between functors is the same as the difference between considering isomorphism of categories and equivalence of categories.

Bicategory	Model category $(\mathcal{C}, \mathcal{W})$
categories, functors, identities	(categories, isomorphisms)
categories, functors, natural isomorphisms	(categories, equivalences)

The moral of the story is that model categories, just like higher categories, can be used to account for notions of 2-morphisms between morphisms in categories (or more specifically 2-isomorphisms), even though the notion of higher morphism is not built into the definition of model category. Relative categories can be thought of as "presentations for higher categories": in the same sense that a monoid or a group can be presented by generators and relations, a relative category $(\mathcal{C}, \mathcal{W})$ presents a "higher category" by means of "generators" $\mathcal{C}$ and "relations" $\mathcal{W}$.

In that sense, model categories can present data of higher dimension than bicategories can: as stated in the homotopy hypothesis (slogan 2.2.19), the Quillen model structure on Top presents the "higher category of ∞-groupoids", which should be nothing less than an ∞-category of some sort. This statement will be made more precise in chapter 6 (see remark 6.4.25). However, only invertible higher morphisms can be recovered through the language of model categories: in the case of the canonical model structure on Cat, we recovered only natural isomorphisms by localizing, and not the more general non-invertible natural transformations. To be more precise we introduce the following terminology.

Definition 2.3.15. In any framework or definition of higher categories, we say a higher category $\mathcal{C}$ is an (n, r)-category when:

- every pair of k-morphisms with same domain and codomain, with $k > n$ are isomorphic,

- every k-morphism for $k > r$ is an isomorphism.

This definition makes sense whenever $r \leq n + 1$. It follows immediately from the definition that whenever $n \leq n', r \leq r'$, any (n, r)-category is also an (n', r')-category.

Example 2.3.16.

- $(0, 0)$-categories are sets,

- $(0, 1)$-categories are posets.

- $(1, 1)$-categories are ordinary categories,

- $(2, 2)$-categories are bicategories (e.g. the bicategory of categories, functors and natural transformations),

- $(2, 1)$-categories are bicategories whose 2-morphisms are invertible (e.g. the bicategory of categories, functors and natural isomorphisms)

- $(1, 0)$-categories are groupoids,

- $(2, 0)$-categories are 2-groupoids,

- $(\infty, 0)$-categories are ∞-groupoids (i.e. , by the homotopy hypothesis, homotopy types).

With that terminology, we can say the following:

Slogan 2.3.17. Relative categories are presentations for $(\infty, 1)$-categories. Model categories are presentations for certain $(\infty, 1)$-categories with nice properties (for instance existence of limits and colimits).

From the point of view that model categories are presentations for $(\infty, 1)$-categories, the adjunction of fact 2.2.18 presents an adjunction between the $(2, 1)$-category of categories and the $(\infty, 1)$-category of homotopy types, and the restriction of this adjunction to an equivalence between the $(2, 1)$-category of groupoids and the $(2, 1)$-category of homotopy 1-types.

2.4   Homotopy theories and ∞-categories

From the point of view of homotopy theorists, any relative category can be thought of as "a homotopy theory", where the homotopy theory of topological spaces and weak homotopy equivalences is referred to as the "classical homotopy theory". Among relative categories, the model categories correspond to some "nice" homotopy theories.

As we have seen, categories and equivalences of categories form a homotopy theory, and as such one would expect that any reasonable definition of higher categories should fit into a homotopy theory. From the previous discussion of relative categories being presentations for $(\infty, 1)$-categories, we saw that somehow, going from $(1, 1)$-categories to $(\infty, 1)$-categories might be a way simpler first step than trying to go directly to (∞, ∞)-categories. Even before that, going from $(1, 0)$-categories (groupoids) or even $(0, 0)$-categories (set) to $(\infty, 0)$-categories (∞-groupoids) was as simple as going from discrete topological spaces (sets) to more general spaces. For the purposes of higher category theory, one of the main takeaways of homotopy theory is that:

Slogan 2.4.1. The combinatorics of higher *invertible* cells is way simpler than the general combinatorics of higher non-invertible cells. In (n, r)-categories, increasing r yields way more complex phenomena than increasing n.

And indeed, we will see in chapter 6 that the category of *simplicial sets* admits a model structure (the so called Joyal model structure) which presents the homotopy theory of $(\infty, 1)$-categories. The theory of $(\infty, 1)$-categories has been thoroughly developed using simplicial sets in the works of André Joyal and Jacob Lurie [Joy08; Lur09; Lur17; Lur18], and we will see in chapter 6 how most of category theory can be done at the level of $(\infty, 1)$-categories, reinforcing the idea that $(\infty, 1)$-categories are conceptually closer to $(1, 1)$-categories than $(2, 2)$-categories are to $(1, 1)$-categories. Because of this, through the rest of this thesis, the term "∞-category" will mean $(\infty, 1)$-category unless otherwise specified. The theory of (∞, n)-categories for $n > 2$ can then be developed from the theory of ∞-categories through iterated weak enrichment [GH15], but we will not use this except in a few remarks at the end of section 7.2.3.

Since relative categories can be thought of as presentations for $(\infty, 1)$-categories, we could expect to be able to formalize that by finding a functor from the category of relative categories to the category of simplicial sets that is compatible with weak equivalences. And indeed such a functor can be constructed, as was shown in 2012 by Clark Barwick and Daniel M. Kan [BK12]. What may come as a surprise is that they showed that this functor is actually an equivalence of homotopy theories: in a sense, every $(\infty, 1)$-category can be obtained as the localization of a category at a class of weak equivalences.

This doesn't mean, however, that working with $(\infty, 1)$-categories is futile: their result actually shows that every $(\infty, 1)$-category can be presented by a *relative poset*; in particular, every category can be obtained as the localization of a poset at a subposet. But clearly, working with categories is very different than working with relative posets: for instance it is completely unclear how to concretely present the category of sets as a localization of a (large) poset. In the same way, having concrete definitions for $(\infty, 1)$-categories allows one to do constructions and definitions directly at the level of $(\infty, 1)$-categories, without having to rely on 1-categorical presentations.

2.5   Homotopy type theory

Although Joyal and Lurie's work has made it possible to work directly with ∞-categories, doing so still requires going through a lot of technicalities on the combinatorics and simplicial sets, and a lot of "low-level" results had to be proven in order for "high-level" users of the theory to be able to handle ∞-categories with the same kind of abstract language they use for categories. Because of this, all of our work in the setting of ∞-categories is postponed to chapter 6, after chapter 4 on our work in *homotopy type theory*, a setting in which homotopical intuitions are closer to the actual formalism.

While all previous discussions were based on set-theoretical foundations for mathematics, homotopy type theory is a completely separate foundational system, in which the primitive objects (types) behave like homotopy types rather than sets. At its heart, homotopy type theory is based on (Martin Löf's) type theory, making it especially approachable to people with more background in computer science than mathematics.

Learning homotopy type theory before the theory of ∞-categories can be very helpful in building intuitions about homotopical concepts before delving into the technicalities of simplicial sets. Indeed, the Joyal-Lurie approach to ∞-categories relies heavily on the theory of model categories, simplicial sets are just one possible model for ∞-categories among many others (relative categories, complete Segal spaces...), whereas in homotopy type theory we are not working with any particular model of homotopy types, the theory is more akin to an *internal language* of homotopy theory, so that every notion discussed in homotopy type theory is *homotopically meaningful*.

2.6   Fibred and indexed points of view

The goal of this section is to present a seemingly simple but very deep idea with which will be of great philosophical and technical importance in the rest of this thesis.

2.6.1   Subsets

Starting from a set X, any subset of X is uniquely characterized by its characteristic function.

Definition 2.6.1. The *characteristic function* of a subset $U \subseteq X$ is the function

$$\mathbf{1}_U : X \to \mathrm{Bool}$$
$$x \mapsto \begin{cases} 1 \text{ if } \in U \\ 0 \text{ if} x \notin U \end{cases}$$

The mapping $U \mapsto \mathbf{1}_U$ defines a bijection

$$P(X) \xrightarrow{\sim} \mathrm{Bool}^X$$

(where $P(X)$ is the powerset of X), with inverse map given by

$$\chi \mapsto \{x \in X \mid \chi(x) = 1\}.$$

Remark 2.6.2. Categorically speaking, the powerset functor $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ (whose action on maps is given by taking inverse images of subsets) is represented by the set Bool.

Moreover, the canonical order on $P(X)$ given by subset inclusion is mapped to the order on Bool^X inherited pointwise from the $0 < 1$ order on Bool, so this is actually an isomorphism of posets.

These two equivalent points of view are referred to as the fibred and indexed points of view on subsets. The word indexed here refers to maps $f : X \to \text{Bool}$ being families of booleans *indexed by X*. To understand where the word *fibred* comes from, let's look at another situation.

2.6.2 Families of sets

Any function $f : Y \to X$ between sets determines a family of sets $(f^{-1}(x))_{x \in X}$ indexed by x. Thinking of this situation with a geometric intuition, the set Y "lives above X" via the function f. Every $x \in X$ determines a "slice " or "*fiber*"of Y" above it: the set $f^{-1}(x)$, and all those fibers put together constitute the whole of the set Y.

Definition 2.6.3. Given $x \in X$, the set $f^{-1}(x)$ is called the *fiber* of f at x.

Reciprocally, taking a family $(Y_x)_{x \in X}$ of sets indexed by X, taking their disjoint union

$$\bigsqcup_{x \in X} Y_x := \{(x, y) \mid x \in X, y \in Y_x\}$$

yields a set with a canonical projection map

$$\bigsqcup_{x \in X} Y_x \to X$$
$$(x, y) \mapsto x$$

This correspondence is not exactly a bijection because taking the disjoint union of the $f^{-1}(x)$ only yields a set *isomorphic* to Y in general. Here category theory helps us get a nicer statement.

Fact 2.6.4. *The operation*
$$(f : Y \to X) \mapsto (f^{-1}(x))_{x \in X}$$

defines a functor
$$\text{Set}_{/X} \to \text{Set}^X$$

from the slice category of Set *over X to the category of functors from X (seen as a discrete category) to* Set.

This functor is an equivalence of categories.

Adapting remark 2.6.2 above gives us the following bicategorical statement:

Remark 2.6.5. The (2-)functors

$$\text{Set}^{\text{op}} \to \text{Cat}$$
$$X \mapsto \text{Set}_{/X}$$
$$X \mapsto \text{Set}^X$$

are naturally isomorphic (in a suitable bicategorical sense). We could almost say that the (2-)functor $\text{Set}_{/(-)}$ is "represented by the category Set ", but that statement does not make sense as is.

Base object	Indexed point of view	Fibred point of view
Set X	map $\chi : X \to \mathrm{Bool}$	subset $U \subseteq X$
Set X	family $(Y_x)_{x \in X}$	fibred set $Y \to X$
Topological space X	continuous map[1] $\chi : X \to \mathbb{S}$	open subset $U \subseteq X$
Topological space X	sheaf $F : O(X)^{\mathrm{op}} \to \mathrm{Set}$	etale space $Y \to X$
Topological space X	locally constant sheaf $F : O(X)^{\mathrm{op}} \to \mathrm{Set}$	covering space $Y \to X$
Groupoid X	presheaf $F : X^{(\mathrm{op})} \to \mathrm{Set}$	discrete (op)fibration $Y \to X$
Category $\mathcal{C}$	presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$	discrete fibration $\mathcal{D} \to \mathcal{C}$
Category $\mathcal{C}$	copresheaf $F : \mathcal{C} \to \mathrm{Set}$	discrete (op)fibration $\mathcal{D} \to \mathcal{C}$
Groupoid X	$F : X \to \mathrm{Gpd}$	Isofibration $Y \to X$
Category $\mathcal{C}$	$F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$	Grothendieck fibration $\mathcal{D} \to \mathcal{C}$
Category $\mathcal{C}$	$F : \mathcal{C} \to \mathrm{Cat}$	Grothendieck opfibration $\mathcal{D} \to \mathcal{C}$

Table 2.1: Various fibred/indexed correspondences.

In more intuitive terms, we just stated that there is a perfect correspondence between families of sets $(Y_x)_{x \in X}$ *indexed* by X, and sets Y equipped with a map $f : Y \to X$. This correspondence is done by

- thinking of the sets Y_x as being a partition into slices, or rather *fibers* of the set Y,

- and thinking of the set Y as the being the putting together of the *fibers* of the map f.

There is no general theorem or statement than can encompass every mathematical situation similar to the two above, so the terms indexed and fibred have more of a philosophical meaning.

Slogan 2.6.6. In many mathematical situations, one can find a correspondence between

1. families of some kind indexed by some mathematical object X,

2. maps $f : Y \to X$ satisfying some property.

In this kind of correspondence, the first point of view is called the *indexed point of view* while the second one is called the *fibred point of view*.

A summary of such situations can be found in table 2.1.

Notation 2.6.7. In fibred situations, given a morphism $f : Y \to X$, we will usually denote the fibers of f as $Y_x := f^{-1}(x)$ when f is clear from the context.

We now focus on two more examples whose ∞-categorical generalizations will play a central role in chapter 6.

---

[1]Here $\mathbb{S}$ denotes the Sierpinski space $\mathbb{S} := \{0, 1\}$ with open subsets $\varnothing$, $\{1\}$ and $\mathbb{S}$.

2.6.3 Isofibrations

Let X be a groupoid and $F : X \to \mathrm{Gpd}$ a functor from X to the category of groupoids. We can define a groupoid $\int_X F$ whose objects are pairs (x, e) with $x \in X$ and $e \in F(x)$, and whose morphisms $(x, e) \to (x', e')$ are given by pairs (f, g) with

$$f : x \to x'$$
$$g : F(f)(y) \to y'$$

The first projection yields a functor $\pi : \int_X F \to X$.

Definition 2.6.8. The operation $(X, F) \mapsto (\int_X F, \pi)$ is called the *Grothendieck construction*. When F takes values in Set, $\int_X F$ is also called the *category of elements* of F.

The projection functor $\pi : \int_X F \to X$ has the following interesting property: given an isomorphism $f : x \xrightarrow{\sim} x'$ in X and an element $(x, e) \in \pi^{-1}(x)$, there exists an element $(x', e') \in \pi^{-1}(x')$ and an isomorphism $(f, g) : (x, e) \xrightarrow{\sim} (x', e')$ that π maps back to f. A functor satisfying this property is called an isofibration.

Definition 2.6.9. A functor $P : \mathcal{D} \to \mathcal{C}$ is said to be an *isofibration* if for every isomorphism $f : x \xrightarrow{\sim} y$ in $\mathcal{C}$ and every object $u \in \mathcal{D}_x$, there exists an object v in $\mathcal{D}_y$ and an isomorphism $g : u \xrightarrow{\sim} v$ in $\mathcal{D}$ such that $P(v) = y$ and $P(g) = f$.

$$
\begin{array}{ccc}
\mathcal{D} & & u \xdashrightarrow{\exists g} \exists v \\
P \downarrow & & \downarrow \qquad \vdots \\
\mathcal{C} & & x \xrightarrow{\ f\ } y
\end{array}
$$

Given an isofibration $P : \mathcal{D} \to \mathcal{C}$ and an isomorphism $f : x \xrightarrow{\sim} y$ in $\mathcal{C}$, the choice for every $u \in \mathcal{D}_x$ of an isomorphism $u \to \sim v$ in $\mathcal{D}$ that maps onto f by P determines a functor $f_! : \mathcal{D}_x \to \mathcal{D}_y$ (read "f shriek"). Making a choice of $f_!$ for every isomorphism f in $\mathcal{C}$, we can find natural isomorphisms

$$(g \circ f)_! \simeq g_! \circ f_!$$

for every pair of composable isomorphisms in $\mathcal{C}$. When $\mathcal{C}$ and $\mathcal{D}$ are groupoids, the mapping

$$x \mapsto \mathcal{D}_x$$
$$f \mapsto f_!$$

assembles into 2-functor $P^{-1}(-) : \mathcal{C} \to \mathrm{Gpd}$ (the term *pseudofunctor* can also be found in the literature, since $(g \circ f)_! \simeq g_! \circ f_!$ is a natural isomorphism rather than an equality of functors). This sort of bicategorical equivalence between functors $\mathcal{C} \to \mathrm{Gpd}$ and isofibrations of groupoids $\mathcal{D} \to \mathcal{C}$ is a first approach to the indexed/fibred correspondence for groupoids.

However, the reader may notice that the notion of isofibration breaks the principle of equivalence (slogan 2.1.1) as its definition relies on equality of objects in a category. As an example, let I denote the *walking isomorphism category*, with objects $0, 1$ and a single isomorphism $f : 0 \xrightarrow{\sim} 1$ between them. The inclusion functor $P : \{0\} \hookrightarrow I$ is clearly not an isofibration, however it is an equivalence of groupoids, so it is isomorphic to the identity of I in the slice bicategory $\mathrm{Gpd}_{/I}$, and the identity functor is clearly an isofibration.

In a sense, the operation of taking the fibers of a functor already breaks the principle of equivalence: here we have $P^{-1}(0) = \{0\}$ while $P^{-1}(1) = \varnothing$ even though 0 and 1 are isomorphic objects in I.

Slogan 2.6.10. A functor is an isofibration precisely if the operation of taking its fibers respects the principle of equivalence.

As it turns out, the isofibrations are precisely the fibrations in the canonical model structure on Cat. This can give an intuition of what fibrations in model categories might look like: they can be thought as morphisms whose "fibers" are homotopically meaningful.

The general theory of model categories states that any morphism $f : x \to y$ in a model category always factors as an *acyclic cofibration* (i.e. a map that's both a cofibration and a weak equivalence) followed by a fibration.

$$x \underset{\sim}{\hookrightarrow} q \underset{F}{\twoheadrightarrow} y$$

In particular this means that every morphism is weakly equivalent to a fibration. In the theory of model categories, this means that fibrations are a purely technical notion: in the higher category represented by a model category, every morphism "is" a fibration (and similarly for cofibrations). In our situation with groupoids, this means that the full sub-bicategory of Gpd $_{/\mathcal{C}}$ on isofibrations is actually equivalent to all of Gpd $_{/\mathcal{C}}$.

To sum this all up in one sentence: fixing a groupoid X, there is an exact bicategorical correspondence between *arbitrary* groupoids over [1] X and families of groupoids indexed by X.

2.6.4 Grothendieck (op)fibrations

While a functor $F : X \to$ Gpd from a groupoid to the category Gpd of groupoids corresponds in the fibred point of view to an isofibration $P : Y \to X$, what about more general functors $F : \mathcal{C} \to$ Cat where $\mathcal{C}$ is a category ? Well, given any such family of categories $F : \mathcal{C} \to$ Cat indexed on $\mathcal{C}$, we can do the same Grothendieck construction as before and obtain a category $\int_{\mathcal{C}} F$ whose

- objects are pairs (x, e) with $x \in \mathcal{C}$ and $e \in F(x)$,

- morphisms $(x, e) \to (x', e')$ are given by pairs (f, g) with $f : x \to x'$ in $\mathcal{C}$ and $g : F(f)(e) \to e'$ in $F(x')$,

and this category comes equipped with a canonical projection functor

$$\pi : \int_{\mathcal{C}} F \to \mathcal{C}.$$

This functor has the property of being a Grothendieck opfibration, for which we first need to introduce the notion of P-cocartesian morphism:

Definition 2.6.11. Let $P : \mathcal{D} \to \mathcal{C}$ be a functor. A morphism $f : x \to y$ in $\mathcal{D}$ is said to be P-*cocartesian* if for every morphism $g : x \to z$ in $\mathcal{D}$ and $w : P(y) \to P(z)$ in $\mathcal{C}$ such that $P(g) = w \circ P(f)$, there exists a morphism $\widehat{w} : y \to z$ in $\mathcal{D}$ such that $P(\widehat{w}) = w$.

---

[1] An "object over x" in a category $\mathcal{C}$ is an object of the slice category $\mathcal{C}_{/x}$, i.e. an object y together with a map $f : y \to x$.

More clearly on a diagram:

$$
\begin{array}{ccc}
\mathcal{D} & & x \xrightarrow{\quad f \quad} y \xdashrightarrow{\exists \widehat{w}} z \quad \overset{\forall g}{\frown} \\[2pt]
P \Big\downarrow & & \\[2pt]
\mathcal{C} & & P(x) \xrightarrow{P(f)} P(y) \xrightarrow{\forall w} P(z)
\end{array}
$$

Definition 2.6.12. A functor $P : \mathcal{D} \to \mathcal{C}$ is said to be a *Grothendieck opfibration* if for every morphism $f : x \to y$ in $\mathcal{C}$ and $u \in \mathcal{D}_x$ (i.e. $u \in \mathcal{D}$ and $P(u) = x$), there exists an object $v \in \mathcal{D}_y$ and a *P-cocartesian* morphism $g : u \to v$ in $\mathcal{D}$ such that $P(g) = f$. In diagrammatic form:

$$
\begin{array}{ccc}
\mathcal{D} & & u \xdashrightarrow{\exists g \ P\text{-cocartesian}} \exists v \\[2pt]
P \Big\downarrow & & \\[2pt]
\mathcal{C} & & x \xrightarrow{\quad f \quad} y
\end{array}
$$

In that case we say that g is a P-*cocartesian lift* of f.

Going from a Grothendieck opfibration $P : \mathcal{D} \to \mathcal{C}$ back to a 2-functor $F : \mathcal{C} \to \mathrm{Cat}$ is done similarly to the case of isofibrations, by making a global choice of P-cocartesian lift for every morphism in $\mathcal{C}$ to obtain functors $f_! : \mathcal{D}_x \to \mathcal{D}_y$ for every $f : x \to y$ in $\mathcal{C}$.

The names *co*cartesian and *op*fibration suggest that there is a dual notion, and indeed when considering categories instead of groupoids, variance comes into play. If we started from a contravariant family of categories $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ indexed on $\mathcal{C}$ instead of a covariant one, we could:

- simply see it as a covariant family on $\mathcal{C}^{\mathrm{op}}$, do the previous Grothendieck construction and obtain a Grothendieck opfibration $\pi : \int_{\mathcal{C}^{\mathrm{op}}} F \to \mathcal{C}^{\mathrm{op}}$ over $\mathcal{C}^{\mathrm{op}}$,

- or we could do a variant of the Grothendieck construction, written $\int_{\mathcal{C}} F$ (even though F has domain $\mathcal{C}^{\mathrm{op}}$), where

  - the objects are pairs (x, e) with $x \in \mathcal{C}$, $e \in F(x)$,
  - the morphisms $(x, e) \to (x, e')$ are pairs (f, g) with $f : x \to x'$ and $g : e \to F(f)(e')$.

Notice how in the variant we had to consider morphisms $g : e \to F(f)(e')$ in the category $F(x)$ instead of the morphisms $g : F(f)(e) \to e'$ in the category $F(y)$ because of the change in variance. This Grothendieck construction also comes equipped with a projection functor $\pi : \int_{\mathcal{C}} F \to \mathcal{C}$, and functors obtained in this way are Grothendieck fibrations, defined using cartesian morphisms.

Definition 2.6.13. Let $P : \mathcal{D} \to \mathcal{C}$ be a functor. A morphism $f : x \to y$ in $\mathcal{D}$ is P-*cartesian* if it satisfies the following unique lifting property.

$$
\begin{array}{ccc}
\mathcal{D} & & z \xdashrightarrow{\exists ! \widehat{w}} x \xrightarrow{\quad f \quad} y \quad \overset{\forall g}{\frown} \\[2pt]
P \Big\downarrow & & \\[2pt]
\mathcal{C} & & P(z) \xrightarrow{\forall w} P(x) \xrightarrow{P(f)} P(y)
\end{array}
$$

Definition 2.6.14. A functor $P : \mathcal{D} \to \mathcal{C}$ is a *Grothendieck fibration* if for every morphism $f : x \to y$ and $u \in \mathcal{D}_x$, there exists a P-cartesian lift of f with domain u.

Remark 2.6.15. A morphism $f : x \to y$ in $\mathcal{D}$ is P-cartesian if and only if it is P^{op}-cocartesian, where $P^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. Similarly, P is a Grothendieck fibration if and only if P^{op} is a Grothendieck opfibration.

Remark 2.6.16. A functor between groupoids is a Grothendieck (op)fibration if and only if it is an isofibration. More generally, every Grothendieck (op)fibration between categories is an isofibration, and the converse is true when the categories involved are groupoids. This is because any cartesian lift of an isomorphism has to be an isomorphism, and reciprocally any isomorphism that lifts an isomorphism is a (co)cartesian lift.

Remark 2.6.17. Just like isofibrations, Grothendieck (op)fibrations do not respect the principle of equivalence. Unlike isofibrations however, not every functor is equivalent to a Grothendieck (op)fibration, as the fibers of a general functor $P : \mathcal{D} \to \mathcal{C}$ do not have to vary covariantly nor contravariantly over $\mathcal{C}$. A "corrected" version of Grothendieck (op)fibrations that respects the principle of equivalence has been introduced by Ross Street in the 80's in his more general study of fibrations in bicategories [Str80; Str81], and are now sometimes called *Street fibrations*.

While we will not directly use Grothendieck (op)fibrations in this thesis, we thought best to introduce them early, as their ∞-categorical versions (called (co)cartesian fibrations of ∞-categories) will play a central role throughout chapter 6, notably to motivate Lurie's very definition of (symmetric) monoidal ∞-categories in section 6.10.

Chapter 3

Linear logic and its categorical semantics

Linear logic was introduced by Jean-Yves Girard in 1987 in his seminal paper "Linear logic" ([Gir87]). In the book [GLT89], he explains that it was his study of coherence space semantics for type theory that led him to discover linear logic: some meaningful constructions apparent in the semantics had no syntactic counterpart, notably what we now know as the exponential modality of linear logic. Even before that, in a 1986 comment in his paper "Normal functors, power series and λ-calculus" ([Gir88]) he explains how with hindsight, his model of λ-calculus based on *normal functors* already implicitly contained all the operations to be found in linear logic.

In this chapter, we begin by giving an exposition of the syntax of linear logic in section 3.1. We then motivate the general notion of categorical semantics in section 3.2 and review its application to linear logic in section 3.3, before presenting in section 3.4 various axiomatizations for categorical models of linear logic that we will encounter throughout the rest of this thesis: Lafont categories, Seely categories, linear categories and linear/non-linear adjunctions.

After setting up all this theory, we review the prototypical model of linear logic, the *relational model*, in section 3.5. In section 3.6, we give an abstract categorical account of what makes this model work, proving novel results on the functoriality of the construction of categories of internal relations: this section is rather technical, but serves as an interesting point of comparison with the analogous results for categories of internal spans we prove in chapter 5, which we motivate in section 3.7.

We then present an alternative point of view on relations — the so called *extensional point of view* — in section 3.8, and through it we review the model of linear logic based on Scott domains in section 3.9. We close the 1-categorical exposition of models of linear logic with models originating in linear algebra in section 3.10, in particular providing exposition for some results regarding exponential comonads on categories of R-modules that may be folklore but don't seem mentioned a lot in the literature.

In section 3.11, we review bicategorical models of linear logic based on profunctors, by analogy with the relational model and the model of domains we surveyed before, paving the way for their ∞-categorical generalizations in section 7.2. We close the chapter with a brief discussion of differential linear logic and its categorical semantics, especially in the bicategorical setting, in section 3.12.

3.1   The syntax of linear logic

Linear logic is a (family of) logical system(s) based around the decomposition of the classical (or intuitionistic) implication connective

$$A \implies B$$

as as linear implication "$\multimap$" together with an exponential modality "!". To explain what this means, we can look at the *structural rules* of contraction and weakening in the sequent calculus presentation of classical (or intuitionistic) logic.

The weakening rule

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \text{ (weak)}$$

states that if we can prove B with hypotheses Γ we can always add an unnecessary hypothesis A and still be able to prove B. In other words, the proof of B is allowed to *discard* the hypothesis A.

The contraction rule

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{ (contr)}$$

states that if we can prove B with hypotheses Γ, A, A, then can also prove B with only the hypotheses Γ, A. In other words, the proof of B is allowed to *duplicate* the hypothesis A.

These rules correspond to the usual philosophical intuition of how to use hypotheses in mathematical reasoning: any hypothesis may be used several times, or none at all. In linear logic however, these rules are thrown away. What this means is that any proof of $\Gamma \vdash A$ in linear logic must use every one of the hypotheses in Γ *exactly once*. Because of this, hypotheses in linear logic may be better understood as representing *resources* used in a formal construction rather than mathematical hypotheses: a resource cannot be duplicated freely, neither can it be discarded.

In linear logic, the standard symbol for implication, $\implies$, is replaced by the symbol $\multimap$ (read "linear implies" or simply "lollipop" due to its shape), to convey the difference in meaning between:

Syntax	Meaning
$A_1, \ldots, A_n \implies B$	B can be proven from the hypotheses $A_1, \ldots, A_n$
$A_1, \ldots, A_n \multimap B$	B can be constructed from the resources $A_1, \ldots, A_n$

In the absence of structural rules, different rules of the sequent calculus for introducing conjunction and disjunction are no longer equivalent, for instance the right rules

$$\frac{\Gamma \vdash A \qquad \Gamma, \vdash B}{\Gamma \vdash A \wedge B} \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta, \vdash B}{\Gamma, \Delta \vdash A \wedge B}$$

are not mutually derivable without contraction and weakening. Because of this, linear logic has two conjunctive connectives, $\otimes$ (read "tensor" or "times") and & (read "with"), and two disjunctive connectives, $\invamp$ (read "par") and $\oplus$ (read "plus"). For instance in linear logic we have the following rules rules:

$$\frac{\Gamma \vdash A \qquad \Gamma, \vdash B}{\Gamma \vdash A \,\&\, B} \;\;(\&\mathrm{R}) \qquad\qquad\qquad \frac{\Gamma \vdash A \qquad \Delta, \vdash B}{\Gamma, \Delta \vdash A \otimes B} \;\;(\otimes\mathrm{R})$$

Intuitively, the $\otimes$ connective can be thought as behaving like the tensor product of vector spaces in linear algebra, while the $\&$ connective behaves like the cartesian product. This intuition can actually be made precise, as intuitionistic linear logic admits models in categories of vector spaces, something we will discuss in section 3.10.

In addition to all of the above, linear logic is also equipped with two modalities ! (read "bang" or "of course") and ? (read "why not"), called the exponentials, that can be used to recover a more intuitionistic or classical behaviour. The idea being that while a general resource A cannot be duplicated or discarded, the resource $!A$ can, and represents "having as much A as one needs" (the modality "?" being, in a sense, dual to "!"). In this thesis we will mostly be concerned about *intuitionistic linear logic* (or ILL for short), in which the connective $\parr$ and the exponential ? are not (usually) considered, and its fragment *MELL* for multiplicative exponential linear logic, where only the connectives $\otimes, \mathbf{1}, \multimap$ and ! are considered. Intuitionistic linear logic is called so because intuitionistic logic admits a translation into it [Gir87, Section 5.1], where the intuitionistic implication $A \implies B$ is translated as

$$!A \multimap B,$$

following the idea that an intuitionistic proof is like a linear proof where resources can be duplicated and discarded.

Even though we will mostly be interested in intuitionistic linear logic, let us mention briefly the full syntax of *classical linear logic* (CLL for short).

In classical linear logic, the grammar of formulae is given by

$$
\begin{aligned}
A ::=\;& X, Y \ldots &&\mid X^{\perp}, Y^{\perp} \ldots \\
&\mid A \otimes A \mid \mathbf{1} \\
&\mid A \parr A \mid \perp \\
&\mid A \,\&\, A \mid \top \\
&\mid A \oplus A \mid \mathbf{0} \\
&\mid !A \mid ?A
\end{aligned}
$$

where $X, Y \ldots$ are *atomic formulae*, and $X^{\perp}, Y^{\perp} \ldots$ represent the negation of atomic formulae. Linear implication is not a primitive operation, instead the operation of negation $(-)^{\perp}$ is extended to an involutive operation on all formulae by induction:

$$
\begin{aligned}
(X)^{\perp} &:= X^{\perp} & (X^{\perp})^{\perp} &:= X \\
(A \otimes B)\perp &:= A^{\perp} \parr B^{\perp} & (\mathbf{1})^{\perp} &:= \perp \\
(A \parr B)^{\perp} &:= A^{\perp} \otimes B^{\perp} & (\perp)^{\perp} &:= \mathbf{1} \\
(A \,\&\, B)^{\perp} &:= A^{\perp} \oplus B^{\perp} & (\top)^{\perp} &:= \mathbf{0} \\
(A \oplus B)^{\perp} &:= A^{\perp} \,\&\, B^{\perp} & (\mathbf{0})^{\perp} &:= \top \\
(!A)^{\perp} &:= ?A^{\perp} & (?A)^{\perp} &:= !A^{\perp}
\end{aligned}
$$

and only then is linear implication defined as

$$A \multimap B \quad := \quad A^{\perp} \,\mathfrak{P}\, B.$$

The connectives $\otimes, \mathbf{1}, \mathfrak{P}, \perp$ are said to be *multiplicative* while $\&, \top, \oplus, \mathbf{0}$ are said to be *additive*, because of how they interact with each other and with the exponentials ! and ? (facts 3.1.3 and 3.1.6).

This presentation has the advantage that it allows for a proof system based on a calculus of one-sided sequents, i.e. sequents of the form $\vdash A_1, \ldots, A_n$. In this system an ordinary sequent $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$ would be represented as $\vdash A_1^{\perp}, \ldots A_m^{\perp}, B_1, \ldots, B_n$.

It's easy to see why this presentation would not be of much use as far as *intuitionistic logic* is concerned: in the intuitionistic sequent calculus, sequents are only allowed to have one formula on the right-hand side (e.g. $A_1, \ldots A_n \vdash B$), and implication plays a primitive role as it cannot be defined in terms of disjunction.

Instead, in intuitionistic linear logic, the grammar of formulae is reduced to:

$$
\begin{aligned}
A ::= \ & X, Y \\
| \ & A \otimes A \mid \mathbf{1} \\
| \ & A \,\&\, A \mid \top \\
| \ & A \oplus A \mid \mathbf{0} \\
| \ & !A
\end{aligned}
$$

excluding the "?" modality and the "multiplicative disjunction". Duality disappears, and sometimes the additive connectives are not even considered, so that we are left with $\otimes, \mathbf{1}, \multimap, !$. Sequents are one-sided with only one formula on the right $A_1, \ldots, A_n \vdash B$, as in the usual sequent calculus for intuitionistic logic.

The rules of the sequent calculus for intuitionistic linear logic are given in table 3.1.

We now mention a few statements that are provable in intuitionistic linear logic, together with some informal idea of their meaning. When A and B are formulae, we write $A \dashv\vdash B$ to say that the sequents $A \vdash B$ and $B \vdash A$ are both derivable. All the following statements are directly quoted from Proposition 1 of G.M. Bierman's thesis [Bie93] and are straightforward exercises in applying the rules of table 3.1.

Fact 3.1.1. *The connectives $\otimes, \&$ and $\oplus$ are symmetric, i.e. the following are derivable in ILL.*

1. $A \otimes B \vdash B \otimes A$

2. $A \,\&\, B \vdash B \,\&\, A$

3. $A \oplus B \vdash B \oplus A$

Fact 3.1.2. *The connectives $\otimes, \&$ and $\oplus$ are unital, with units respectively $\mathbf{1}, \top$ and $\mathbf{0}$:*

1. $A \otimes \mathbf{1} \dashv\vdash A$

2. $A \,\&\, \top \dashv\vdash A$

3. $A \oplus \mathbf{0} \dashv\vdash A$

are derivable in ILL.

Fact 3.1.3. *The multiplicative disjunction $\otimes$ distributes over the additive connectives $\&$ and $\oplus$:*

$$\frac{}{A \vdash A} \text{ (axiom)} \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C} \text{ (cut)}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \text{ (exchange)}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \text{ ($\otimes$L)} \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \text{ ($\otimes$R)}$$

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \mathbin{\&} A_2 \vdash B} \text{ ($\&$L}_i) \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B} \text{ ($\&$R)}$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} \text{ ($\oplus$L}_i) \qquad\qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A \oplus B} \text{ ($\oplus$R}_i)$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \text{ ($\multimap$L)} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \text{ ($\multimap$R)}$$

$$\frac{\Gamma \vdash A}{\Gamma, \mathbf{1} \vdash A} \text{ ($\mathbf{1}$L)} \qquad\qquad \frac{}{\vdash I} \text{ ($\mathbf{1}$R)}$$

$$\frac{}{\Gamma, \mathbf{0} \vdash A} \text{ $\mathbf{0}$L} \qquad\qquad \frac{}{\Gamma \vdash \top} \text{ $\top$R}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \text{ (\emph{dereliction})} \qquad\qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \text{ (promotion)}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \text{ (contraction)} \qquad\qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \text{ (weakening)}$$

Table 3.1: Sequent calculus rules for intuitionistic linear logic.

1. $A \otimes (B \,\&\, C) \vdash (A \otimes B) \,\&\, (A \otimes C)$

2. $A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$

are derivable in ILL. This distributivity is one of the reasons to call $\otimes$ *a* multiplicative *connective, and the other two* additive *connectives.*

Remark 3.1.4. Notice how the distributivity of $\otimes$ over & is not an equivalence. From the point of view of resources, $A \otimes B$ can be thought of as "having both A and B at your disposal", while $A \,\&\, B$ should be thought of as "having A and B at your disposal but only being able to use one of them". For instance, if 1€ can buy a baguette and 1€ can buy a croissant, we would have

$$1\text{€} \vdash \text{baguette} \,\&\, \text{croissant}$$

but not

$$1\text{€} \vdash \text{baguette} \otimes \text{croissant},$$

as that would require 2€.

Coming back to the distributivity of $\otimes$ over &, the formula

$$(A \otimes B) \,\&\, (A \otimes C),$$

when used, requires a choice between $(A \otimes B)$ and $(A \otimes C)$, while the formula

$$A \otimes (B \,\&\, C)$$

contains a readily accessible resource A, allowing its user to keep the choice between B and C for later. Intuitively, the latter is stronger than the former, and so both are not equivalent.

Fact 3.1.5. *The exponential of A, is discardable and duplicable, and "contains" A:*

1. $!A \vdash \mathbf{1}$

2. $!A \vdash !A \otimes !A$

3. $!A \vdash A$

are derivable in ILL. In particular, by iterating the duplication we can show $!A \vdash A \otimes \cdots \otimes A$ *for as many copies of A as needed, corresponding to the intuition of* $!A$ *as representing "having as much A as one needs".*

Fact 3.1.6. *The exponential* ! *maps the additive* & *to the multiplicative* $\otimes$:

- $!(A \,\&\, B) \dashv\vdash !A \otimes !B$

- $!\top \dashv\vdash \mathbf{1}$

This property is one reason why the exponential ! *is called so, as it is reminiscent of some defining properties of exponential functions in calculus:* $e^{x+y} = e^x \times e^y$ *and* $e^0 = 1$.

Remark 3.1.7. I'd like to advise any reader interested in gaining more intuition on basic syntactic aspects of linear logic to read Adele Lopez's excellent and very pedagogical blog post *Visual Linear Logic* [Lopb], in which she sketches a visual calculus based on Peirce's "alpha diagrams" to explain the meaning of the connectives and rules of (classical) linear logic.

Beyond the logical content given by the grammar of formulae and the derivation rules, intuitionistic linear logic also enjoys a computational content given by cut-elimination rules. We will not recall here a complete list of cut-elimination rules, redirecting the interested reader to P.A. Melliès's survey [Mel09, Chapter 3] instead. However, we will explain the consequences of these rewrite rules for the categorical semantics of intuitionistic linear logic in section 3.3.

3.2   Categorical semantics

In *denotational semantics* (as opposed to e.g. operational semantics), we are concerned with interpreting the "result" or *denotation* of programs and proofs. That is, *what* the program (or proof) computes rather than *how* it computes. In programming semantics, this means that two programs that *evaluate* to the same result should be interpreted in the same way in denotational models: denotation is an *invariant of β-equivalence*. Through the Curry-Howard correspondence, the process of evaluation corresponds to the *cut-elimination* procedure. Hence denotations should be *invariant under cut-elimination*.

   Categorical semantics is the study of denotational models in categories. As a motivation, let's consider an extremely simple proof system called *simple logic*, with only primitive formulae and no connectives

$$F ::= X, Y, \ldots,$$

two-sided sequents with one formula on each side

$$X \vdash Y$$

and just the axiom and cut rules

$$\frac{}{X \vdash X} \text{ (ax)} \qquad\qquad \frac{X \vdash Y \qquad Y \vdash Z}{X \vdash Z} \text{ (cut)}$$

and a cut-elimination procedure to specify how (axiom) interacts with (cut), as well as how (cut) interacts with itself (see table 3.2). As one can see, the rule $\rightsquigarrow_\alpha$ does not really "eliminate" any

$$\frac{\dfrac{}{X \vdash X} \text{ (ax)} \quad \dfrac{\pi}{X \vdash Y}}{X \vdash Y} \text{ (cut)} \quad \rightsquigarrow_\lambda \quad \dfrac{\pi}{X \vdash Y}$$

$$\frac{\dfrac{\pi}{X \vdash Y} \quad \dfrac{}{Y \vdash Y} \text{ (ax)}}{X \vdash Y} \text{ (cut)} \quad \rightsquigarrow_\rho \quad \dfrac{\pi}{X \vdash Y}$$

$$\frac{\dfrac{\dfrac{\pi_1}{X \vdash Y} \quad \dfrac{\pi_2}{Y \vdash Z}}{X \vdash Z} \text{ (cut)} \quad \dfrac{\pi_3}{Z \vdash T}}{X \vdash T} \text{ (cut)} \quad \rightsquigarrow_\alpha \quad \dfrac{\dfrac{\pi_1}{X \vdash Y} \quad \dfrac{\dfrac{\pi_2}{Y \vdash Z} \quad \dfrac{\pi_3}{Z \vdash T}}{Y \vdash T} \text{ (cut)}}{X \vdash T} \text{ (cut)}$$

Table 3.2: Cut-elimination rules for simple logic.

cuts, but rather makes sure that all cuts will be "pushed to the right". As such it is usually called *commuting conversion* rule rather than a cut-elimination one [Mel09, Section 3.4], but we do not make this technical distinction here.

   Let's say we want to study denotational semantics for simple logic using set theory. To that end we need the following:

   1. a set of *interpretations for formulae*, suggestively named Ob,

2. for every formula X, an element $[\![X]\!] \in \mathrm{Ob}$ called its *interpretation* or *denotation*,

3. for every sequent $X \vdash Y$ a set of *interpretation for proofs of $X \vdash Y$*, suggestively named $\mathrm{Hom}(X, Y)$,

4. for every proof π of $X \vdash Y$, an element $[\![\pi]\!] \in \mathrm{Hom}(X, Y)$ also called its interpretation or denotation,

5. such that whenever $\pi_1 \rightsquigarrow \pi_2$, their denotations are equal: $[\![\pi_1]\!] = [\![\pi_2]\!]$.

Given a formula X, the axiom rule gives a proof of $X \vdash X$, whose interpretation we can call $\mathrm{id}_X \in \mathrm{Hom}(X, X)$. Given proofs $\pi_1 : X \vdash Y$ and $\pi_2 : Y \vdash Z$, the cut rule gives a proof of $X \vdash Z$, whose interpretation we write $[\![\pi_1; \pi_2]\!] \in \mathrm{Hom}(X, Z)$. The invariance by cut-elimination enforces the following equations:

$$\rightsquigarrow_\lambda \quad [\![\mathrm{id}_X; \pi]\!] = [\![\pi]\!]$$
$$\rightsquigarrow_\rho \quad [\![\pi; \mathrm{id}_X]\!] = [\![\pi]\!]$$
$$\rightsquigarrow_\alpha \quad [\![(\pi_1; \pi_2); \pi_3]\!] = [\![\pi_1; (\pi_2; \pi_3)]\!]$$

At this point, it is hard not to recognize the apparent category-like structure. For these axioms to truly give a category, we moreover need:

1. that the set $\mathrm{Hom}(X, Y)$ depends only the denotations $[\![X]\!]$ and $[\![Y]\!]$ of X and Y, so that we can write $\mathrm{Hom}([\![X]\!], [\![Y]\!])$,

2. that the element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ only depends on the denotation $[\![X]\!]$ of X, so that we can write $\mathrm{id}_{[\![X]\!]} \in \mathrm{Hom}([\![X]\!], [\![X]\!])$,

3. that the element $[\![\pi_1; \pi_2]\!]$ only depends on $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$, so that we can write $[\![\pi_1]\!]; [\![\pi_2]\!]$.

Under these 3 additional assumptions, denotational models of simple logic are exactly given by categories.

Remark 3.2.1. In his extensive survey "Categorical semantics of linear logic", Paul-André Melliès calls these additional assumptions *modularity assumptions*, and he notes that modularity and invariance under cut-elimination are what characterize categorical semantics of logic [Mel09, Section 2.1].

Slogan 3.2.2. *Categorical semantics* is the study of denotational semantics of logic in categories: formulae are interpreted by objects, proofs by morphisms, and cut elimination by equality of morphisms.

As we have seen, the simplest setting for categorical semantics is the setting of simple logic. As we add more connectives and cut-elimination rules to the logic, they need to be interpreted by additional structures on categories. For instance, if we now consider multiple formulae on the left-hand side of sequents

$$A_1, \ldots, A_n \vdash B$$

the interpretation will be done in multicategories (also called colored operads), a variant of categories where hom-sets $\mathrm{Hom}(X, Y)$ are allowed to have multiple inputs at once, typically written $\mathrm{Hom}(X_1, \ldots, X_n; Y)$. Depending on the presence of exchange rules

$$\frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \ \mathrm{ex}$$

the multicategories might be planar or non-planar, i.e. $\mathrm{Hom}(\mathbb{X}, Y)$ might be defined either for $\mathbb{X}$ a list of objects or a multiset of objects. We now turn to the categorical semantics of intuitionistic linear logic.

Logic	Categorical semantics
Simple logic $A \vdash B$	Categories
Simple logic $A_1, \ldots, A_n \vdash B$	Multicategories
Simple logic with $\otimes$ and 1	(Symmetric) monoidal categories
Simple logic with $\&$ and $\top$	Cartesian categories
Simply typed λ-calculus	Cartesian closed categories
Dependent type theory	Locally cartesian closed categories
$(\otimes, 1, \multimap)$ fragment of ILL	Symmetric monoidal closed categories
Intuitionistic linear logic	Linear categories
ILL with $\&$ and $\top$	Seely categories

Table 3.3: Categorical semantics for different logical systems.

3.3 Elements of categorical semantics for intuitionistic linear logic

In this section we give general ideas of how the different connectives of ILL should be interpreted in categories, closely following [Mel09, Chapter 2], before presenting more complete axiomatizations in the next section.

Sequents. In intuitionistic linear logic, sequents are of the form

$$A_1, \ldots, A_n \vdash B.$$

As explained in the previous section, this means that proofs in ILL should be interpreted as morphisms in a multicategory rather than a category. Luckily, the left rule for the tensor

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \ (\otimes \text{L})$$

is *invertible*, i.e. whenever any proof of its conclusion implies that there exists a proof for its premise. In particular, this means that proofs of

$$A_1, \ldots, A_n \vdash B$$

correspond bijectively to proofs of

$$A_1 \otimes \cdots \otimes A_n \vdash B,$$

so we can avoid the use of multicategories by interpreting proofs of $A_1, \ldots, A_n \vdash B$ as morphisms

$$\text{Hom}(\llbracket A_1 \otimes \cdots \otimes A_n \rrbracket, \llbracket B \rrbracket).$$

Hence in what follows, we will fix a category $\mathcal{C}$, with set of objects $\text{Ob}\,\mathcal{C}$, and see what structures on $\mathcal{C}$ are needed to model ILL.

The tensor and 1.   The tensor connective $\otimes$ must be interpreted as a function $[\![\otimes]\!] : \operatorname{Ob} \mathcal{C} \times \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{C}$. Moreover, the right rule for the tensor

$$\frac{\Gamma \vdash A \qquad \Delta, \vdash B}{\Gamma, \Delta \vdash A \otimes B} \; (\otimes \mathrm{R})$$

suggests that $[\![\otimes]\!]$ should actually be a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and indeed the cut-elimination rules show that to be the case. Moreover, the derivability of the sequents

$$(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)$$
$$A \otimes \mathbf{1} \vdash A$$
$$A \otimes B \vdash B \otimes A$$

together with other cut-elimination rules imply that $\otimes$ should actually be interpreted as a *symmetric monoidal structure* on $\mathcal{C}$, with unit $[\![\mathbf{1}]\!]$. To avoid cluttering, we will abuse notation and also write $\otimes$ for the functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ instead of $[\![\otimes]\!]$, and similarly $\mathbf{1} \in \operatorname{Ob} \mathcal{C}$ for its unit.

Linear implication.   The right rule for linear implication

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \; (\multimap \mathrm{R})$$

is invertible. Together with the invertibility of the left rule for the tensor, this means there is a correspondence between proofs of $A \otimes B \vdash C$ and $A \vdash B \multimap C$. Together with cut-elimination rules, this can be shown to imply that $B \multimap C$ should be interpreted as an internal hom object from B to C in $\mathcal{C}$. In other words, for every $X \operatorname{in} \operatorname{Ob} \mathcal{C}$, the functor $(X \otimes -) : \mathcal{C} \to \mathcal{C}$ should have a right adjoint, written $(X \multimap -)$, which means there should be a natural bijection

$$\operatorname{Hom}(X \otimes Y, Z) \simeq \operatorname{Hom}(X, Y \multimap Z).$$

This means $\mathcal{C}$ should be a *closed* symmetric monoidal category.

The additive connectives.   Like the tensor, the additive connectives can be interpreted as symmetric monoidal structures on $\mathcal{C}$. However, they must satisfy additional hypotheses: for instance for the & connective, the sequents

$$A \mathbin{\&} B \vdash A$$
$$A \mathbin{\&} B \vdash B$$

are derivable in ILL. Categorically, this suggests that $[\![A \mathbin{\&} B]\!]$ should be interpreted a cartesian product of $[\![A]\!]$ and $[\![B]\!]$ in $\mathcal{C}$, and indeed the cut-elimination rules will enforce that property. Similarly, the connective $\oplus$ needs to be interpreted as a coproduct in $\mathcal{C}$, and the units $\top$ and $\mathbf{0}$ must respectively be interpreted as a terminal object and an initial object. This has the advantage that having (co)products and co(terminal) objects is a property rather than a structure.

So far, we thus need a closed symmetric monoidal category $\mathcal{C}$ *with finite products and coproducts*.

The *exponential*.   The exponential is clearly the most complex connective to interpret. First of all, the derivation

$$\frac{\dfrac{A \vdash B}{!A \vdash B} \; (\text{dereliction})}{!A \vdash !B} \; (\text{promotion})$$

shows that ! should be interpreted as a functor $! : \mathcal{C} \to \mathcal{C}$.

Moreover, from the derivations

$$\frac{\overline{A \vdash A} \ \text{(axiom)}}{!A \vdash A} \ \text{(dereliction)} \qquad\qquad \frac{\overline{!A \vdash !A} \ \text{(axiom)}}{!A \vdash !!A} \ \text{(promotion)}$$

we see that we need natural morphisms $!X \to X$ and $!X \to !!X$. The cut-elimination rules actually show that this needs to define the counit and comultiplication of a comonad structure on !.

Remark 3.3.1. The rules of dereliction and promotion can already be seen as the comonadic versions of the *lift* and *bind* operators for monads as they are usually presented in functional programming languages. The way we recover the counit and comultiplication morphisms from these operations is formally dual to the way monad laws can be recovered from a lift and a bind.

But ! being a comonad on $\mathcal{C}$ is not enough, indeed using contraction and weakening we can derive other sequents like

$$!A \vdash !A \otimes !A$$
$$!A \vdash \mathbf{1}$$

which (as always) together with cut-elimination rules can be shown to imply that every object of the form $!X$ in $\mathcal{C}$ should be equipped with a natural structure of commutative comonoid.

As mentioned in fact 3.1.6, we also have an equivalence $!(A \,\&\, B) \dashv\vdash !A \otimes !B$ which should correspond to natural isomorphisms $!(X \times Y) \simeq !X \otimes !Y$, suggesting that ! should be a (strongly) symmetric monoidal functor from the cartesian structure to the monoidal structure. Additionally, we can derive $!(A \otimes B) \vdash !A \otimes !B$, suggesting that ! should be a lax monoidal functor from the monoidal structure to itself.

The goal of the previous discussion is to give the reader a sense of the difficulty of writing down in detail all the structures involved in the categorical interpretation of the exponential modality !, and even more so trying to find a minimal presentation of all this structure. Thankfully, since the birth of linear logic, many *axiomatizations* have been proposed that give sufficient conditions and structures to interpret intuitionistic linear logic in a category.

3.4 Axiomatizations for categorical models of ILL

In this section we briefly survey the various axiomatizations for categorical models of the exponential of intuitionistic linear logic that have been developed in the 80'-90'. More extensive surveys include [Pai14; Mel03; Mel09].

Lafont categories. The first axiomatization for the exponential of intuitionistic linear logic was proposed in 1988 by Yves Lafont in his thesis [Laf88]. As explained in the previous section, every object $!X$ must come equipped with a natural commutative comonoid structure. Lafont noticed that taking $!X$ to be the *cofree commutative comonoid* cogenerated by X for every object X was enough to model all the denotational properties of the exponential !, leading to the following definition.

Definition 3.4.1. A *Lafont category* is a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \multimap)$ such every object $A \in \mathcal{L}$ admits a cofree commutative comonoid $!A$.

Fact 3.4.2. *Every Lafont category $\mathcal{C}$ defines a model of the MELL fragment of ILL, where the exponential $!X$ of an object X is interpreted as the cofree commutative comonoid on X. In that case we say that $!$ is a free exponential.*

If $\mathcal{C}$ moreover has finite products (respectively coproducts), it is also a model for the additive connective $\&$ (respectively $\oplus$).

In more detail, the *cofree commutative comonoid* on an object X is a commutative comonoid $!X$ together with a map $\eta :!X \to X$ such that for every comonoid M, every map $f : M \to X$ factors uniquely through η as a comonoid morphism $\widehat{f} : M \to !X$:

$$M \xdashrightarrow{\widehat{f}} !X \xrightarrow{\eta} X$$
$$f$$

The existence of a cofree commutative comonoid on every object implies that the forgetful functor

$$L : \mathrm{Comon}\,(\mathcal{C}) \to \mathcal{C}$$

from the category $\mathrm{Comon}\,(\mathcal{C})$ of commutative comonoids in $\mathcal{C}$ to $\mathcal{C}$ admits a right adjoint $R : \mathcal{C} \to \mathrm{Comon}\,(\mathcal{C})$, the *cofree construction*.

$$\mathrm{Comon}\,(\mathcal{C}) \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$$

By definition we have $!X := LR(X)$ for every object $X \in \mathcal{C}$, and the composite functor $LR : \mathcal{C} \to \mathcal{C}$ inherits from the adjunction a comonad structure, which models the comonad laws of the exponential $!$.

Seely categories. In 1989, Robert Seely proposed a more general axiomatization for the $(\otimes, \mathbf{1}, \multimap, \&, !)$ fragment of ILL [See89]. In particular, his axiomatization encompasses the coherence spaces model of linear logic, where the exponential is not free. However, in 1993, Gavin Bierman showed in his thesis that Seely's axiomatization was not sound with respect to ILL [Bie93; Bie95]. He managed to fix the definition by requiring an additional axiom, and named the resulting structure "new-Seely categories". Here we use the name Seely category for the corrected definition, following the presentation of [Mel09].

Definition 3.4.3. A *Seely category* is

1. a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1, \multimap)$

2. with finite products ($\&$ and $\top$),

3. a comonad $(!, \delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$,

4. isomorphisms $m_{A,B}^2 : !(A \,\&\, B) \simeq !A \otimes !B$ and $m^0 : !\top \simeq 1$, called the *Seely isomorphisms*, that make $! : (\mathcal{C}, \&) \to (\mathcal{C}, \otimes)$ into a *symmetric monoidal functor*

5. such that the following diagram commutes (*Bierman's axiom*)

$$
\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\delta_A \otimes \delta_B} & !!A \otimes !!B \\
{\scriptstyle m_{A,B}^2}\downarrow & & \downarrow{\scriptstyle m_{!A,!B}^2} \\
!(A \,\&\, B) \xrightarrow{\delta_{A\&B}} !!(A \,\&\, B) & \xrightarrow[!\langle !\pi_1, !\pi_2\rangle]{} & !(!A \,\&\, !B)
\end{array}
$$

Fact 3.4.4. *Every Seely category $\mathcal{C}$ defines a model of the $(\otimes, \mathbf{1}, \multimap, \&, \top, !)$ fragment of ILL. If $\mathcal{C}$ admits finite coproducts, it also models the additive connectives $(\oplus, \mathbf{0})$.*

Remark 3.4.5. Seely's axiomatization is made to ensure that the Kleisli[1] category $\mathcal{C}_!$ for the comonad ! is cartesian closed. The Kleisli category $\mathcal{C}_!$ has the same objects as $\mathcal{C}$, and the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}_!}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(!X, Y),$$

so morphisms in the Kleisli category correspond to non-linear proofs in ILL, or in other words to intuitionistic proofs, and cartesian closed categories are precisely models of the $(\wedge, \implies)$ fragment of intuitionistic logic.

Remark 3.4.6. Bierman's axiom (item 5 in the definition) is equivalent to asking the canonical forgetful functor

$$\mathcal{C}_! \to \mathcal{C}$$

to be a symmetric monoidal functor from the cartesian structure on $\mathcal{C}_!$ to the tensor structure on $\mathcal{C}$.

Linear categories. In his thesis [Bie93], Bierman shows that his correction to Seely's definition is sound by relating it to another axiomatization he and Benton, de Paiva and Hyland introduced in under the name of linear categories [Ben+92].

Definition 3.4.7. A *linear category* is

1. a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \multimap)$,

2. together with a *lax symmetric monoidal comonad* $((!, m), \delta, \varepsilon)$,

3. and a natural commutative comonoid structure

$$d_A : !A \to !A \otimes !A,$$
$$e_A : !A \to 1,$$

such that d_A and e_A are coalgebra morphisms for ! and δ is a comonoid morphism.

Fact 3.4.8. *Every linear category $\mathcal{C}$ defines a model of the MELL fragment of ILL. Like Lafont categories, $\mathcal{C}$ moreover models $(\&, \top)$ if it admits finite products, and $(\oplus, \mathbf{0})$ if it admits finite coproducts.*

Remark 3.4.9. Given a linear category $\mathcal{C}$, the forgetful functor $\mathcal{C}^! \to \mathcal{C}$ from the category of coalgebras[2] for the comonad ! to $\mathcal{C}$ is a symmetric monoidal functor.

Linear-non-linear adjunctions. A common feature of Lafont categories, Seely categories and linear categories is that all three of them induce an adjunction between a cartesian (closed) category (i.e. a model of intuitionistic logic) and a symmetric monoidal closed category (i.e. a model of $(\otimes, \multimap, \mathbf{1})$). Intuitively, this means that the exponential comonad could be better understood not as a primary construct, but as being induced by an adjunction between a "linear world" and a "non-linear world".

Following this intuition, Benton described in 1994 a logical system with two kinds of formulae

---

[1] We use the terminology of "Kleisli category" for a comonad and not "coKleisli", as the "co" would be redundant with that of the *co*monad. For the same reason we speak of commutative comonoids and not cocommutative comonoids.

[2] Also called Eilenberg-Moore category in the literature.

- the "linear formulae" $A, B, \ldots$,

- the "non-linear formulae" $X, Y, \ldots$

and two kinds of sequents:

- the "linear sequents" $X_1, \ldots, X_n; A_1, \ldots, A_n \vdash_{\mathcal{L}} B$,

- the "non-linear sequents" $X_1, \ldots, X_n \vdash Y$,

together with connectives to pass formulae from the linear world to the non-linear world and vice versa [Ben94]. In his system, both the linear implication $\multimap$ and the intuitionistic implication $\implies$ connectives exist as primitive constructs. Moreover, ILL embeds into the "linear world" of this "mixed linear/non-linear logic", in a way that the exponential is translated into the back-and-forth between the "linear world" and the "non-linear world".

With this syntax, he also introduced a notion of categorical model given by linear/non-linear adjunctions.

Definition 3.4.10. A *linear/non-linear adjunction*, or LNL adjunction for short, is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$$

between a cartesian category $(\mathcal{M}, \times)$ and a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \multimap)$, where the left adjoint $L : \mathcal{M} \to \mathcal{L}$ is strongly monoidal $L(X \times Y) \simeq LX \otimes LY$.

$\mathcal{M}$ is called the *non-linear category* or multiplicative category of the adjunction, and $\mathcal{L}$ is its *linear category*. L is the *linearization functor* and M is the *multiplication functor*.

The apparent clash in naming between the term linear category in definition 7.1.7 and the same term in definition 3.4.7 is solved by the following:

Fact 3.4.11. *In an LNL adjunction*

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$$

the comonad $LM : \mathcal{L} \to \mathcal{L}$ endows $\mathcal{L}$ with the structure of a linear category is the sense of definition 3.4.7.

Fact 3.4.12. *All of the previous axiomatizations induce linear/non-linear adjunction in the following ways:*

Base structure	*Induced LNL adjunction*
Lafont category $(\mathcal{C}, \otimes, \ldots)$	$(\mathrm{Comon}\,(\mathcal{L}), \times) \underset{}{\overset{}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$
Seely category $(\mathcal{C}, \otimes, !, \ldots)$	$(\mathcal{C}_!, \times) \underset{}{\overset{}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$
Linear category $(\mathcal{C}, \otimes, !, \ldots)$	$(\mathcal{C}^!, \times) \underset{}{\overset{}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$

Remark 3.4.13. In an LNL adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\perp}{\rightleftarrows}}} (\mathcal{L}, \otimes)$$

the actual model of ILL is the linear category $\mathcal{L}$ with the exponential comonad given by ! := LM : $\mathcal{L} \to \mathcal{L}$. In particular, different choices of $\mathcal{M}, L$ and M may lead to the same model of linear logic. In that sense, linear/non-linear adjunctions are models of more than just linear logic, they are models for the mixed linear/non-linear logic described by Benton [Ben94], which contains ILL as a fragment.

As mentioned in remark 3.4.6, a Seely category can also be defined to be a symmetric monoidal closed category $\mathcal{C}$ that admits finite products together with a comonad ! : $\mathcal{C} \to \mathcal{C}$ and the structure of a symmetric monoidal functor on the canonical functor $\mathcal{C}_! \to \mathcal{C}$. In other words, a Seely category *is* an LNL adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\rightleftarrows}} (\mathcal{L}, \otimes)$$

where $\mathcal{M}$ is the Kleisli category on $\mathcal{L}$ for the comonad LM.

Now let $\mathcal{L}$ be a linear category. It induces an LNL adjunction with its category of coalgebras $\mathcal{L}^!$. The Kleisli category $\mathcal{L}_!$ is equivalent to the full subcategory of $\mathcal{L}^!$ on cofree coalgebras, i.e. coalgebras of the form !X, so the LNL adjunction can be restricted to an LNL adjunction between $\mathcal{L}$ and $\mathcal{L}_!$. In particular, we have:

Fact 3.4.14. *Any linear category with finite products is a Seely category.*

However this abstract reasoning does not really make clear how to recover the Seely isomorphisms from the lax monoidal structure on $\mathcal{L}$, or more generally how LNL adjunctions recover all the kinds of structures induced by the comonad ! we described in section 3.3. So we now explore how to recover some of these structures through examples.

Let

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\rightleftarrows}} (\mathcal{L}, \otimes)$$

be a linear/non-linear adjunction.

Fact 3.4.15. *Let X be an object of $\mathcal{L}$. Then $LM(X)$ inherits a canonical structure of commutative comonoid.*

This fact relies on the following folklore lemma.

Lemma 3.4.16. *In a category $\mathcal{C}$ with finite products, every object admits a unique commutative comonoid structure with respect to the cartesian product.*

Proof idea. Let X be an object of $\mathcal{C}$. There is a canonical choice of comultiplication given by the diagonal map $\Delta_X : X \to X \times X$, and a canonical choice of counit given by the unique map to the terminal object $\langle \rangle : X \to 1$. It is a straightforward exercise to check that this indeed defines an associative, commutative and unital law.

Now consider another commutative comonoid structure on X, with comultiplication $\mu : X \to X \times X$ and counit $\eta : X \to 1$. Since 1 is a terminal object, there is a unique map from X to 1, so we must have $\eta = \langle \rangle$. Now by unitality of μ, the following diagram must commute:

in particular, $\pi_1 \circ \mu = \text{id}_X$ and $\pi_2 \circ \mu = \text{id}_X$, so by definition of Δ_X we must have $\mu = \Delta_X$, which concludes the proof of unicity. $\qquad\square$

Proof of fact 3.4.15. Let X be an object of $\mathcal{L}$. Then $M(X)$ is an object of the cartesian category $\mathcal{M}$, and hence carries a unique commutative comonoid structure. Since $L : \mathcal{M} \to \mathcal{L}$ is a symmetric monoidal functor, it preserves commutative comonoids, hence $LM(X)$ inherits a commutative comonoid structure from the one on $M(X)$. $\qquad\square$

Fact 3.4.17. *Now suppose $\mathcal{L}$ is moreover cartesian. Then the comonad $! := LM : \mathcal{L} \to \mathcal{L}$ is a symmetric monoidal functor from the cartesian structure on $\mathcal{L}$ to the tensorial structure on $\mathcal{L}$.*

Proof. The functor $M : \mathcal{L} \to \mathcal{M}$ is a right adjoint, so it preserves any limit that exists in $\mathcal{L}$. Since $\mathcal{L}$ is cartesian, M preserves finite products, so it admits a canonical structure of symmetric monoidal functor $(\mathcal{L}, \times) \to (\mathcal{M}, \times)$. Since L is symmetric monoidal from $(\mathcal{M}, \times)$ to $(\mathcal{L}, \otimes)$, the composite functor LM is symmetric monoidal from $(\mathcal{L}, \times)$ to $(\mathcal{L}, \otimes)$. $\qquad\square$

Classical linear logic and $*$-autonomy. All the axiomatizations we have considered so far are axiomatizations for (fragments of) intuitionistic linear logic. There is a simple way to extend any one of these axiomatizations to a model of classical linear logic, using the notion of $*$-*autonomy*.

Definition 3.4.18. Let $(\mathcal{C}, \otimes, \mathbf{1}, \multimap)$ be a symmetric monoidal category (i.e. a model of the $(\otimes, \mathbf{1}, \multimap)$ fragment of ILL). An object $\bot \in \mathcal{C}$ is said to be *dualizing* if for every object $X \in \mathcal{C}$, the canonical morphism

$$\partial_X : X \to (X \multimap \bot) \multimap \bot$$

is an isomorphism.

Here the "canonical morphism" $\partial_X : X \to (X \multimap \bot) \multimap \bot$ is the image through the bijection

$$\text{Hom}(X \otimes (X \multimap \bot), \bot) \simeq \text{Hom}(X, (X \multimap \bot) \multimap \bot)$$

given by the adjunction $(X \otimes -) \dashv (X \multimap -)$ of the evaluation morphism

$$\varepsilon_\bot^X : X \otimes (X \multimap \bot) \to \bot,$$

i.e. the counit of the same adjunction.

Definition 3.4.19. A symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \multimap)$ equipped with a dualizing object $\bot$ is said to be $*$-autonomous (read "star autonomous").

Fact 3.4.20. *A $*$-autonomous category $\mathcal{C}$ is a model of the $(\otimes, \mathbf{1}, \invamp, \bot, \multimap)$ fragment of classical linear logic, where $\bot$ is interpreted as the dualizing object $\bot$, and the $\invamp$ connective is interpreted as*

$$[\![A \invamp B]\!] := \big(([\![A]\!] \multimap \bot) \otimes ([\![B]\!] \multimap \bot)\big) \multimap \bot$$

Moreover, if $\mathcal{C}$ is equipped with a model of the exponential $!$ in the sense of Lafont, Seely, linear categories or LNL adjunctions, then it is also a model of the ? via

$$[\![?A]\!] := (!([\![A]\!] \multimap \bot)) \multimap \bot$$

3.5 The relational model

The simplest and most studied model of linear logic is the relational model. Rather than just reminding basic definitions, we choose in this section to put an emphasis on the indexed and fibred approaches to these usual definitions, following the philosophy of section 2.6.

Definition 3.5.1 (Category of relations, fibred point of view)**.** The category Rel of sets and relations is defined as follows:

- its objects are sets $X, Y, \ldots,$

- a morphism from X to Y is given by a relation $R \subseteq X \times Y$,

- the identity $\mathrm{id}_X \subseteq X \times X$ is given by $\mathrm{id}_X := \{(x, x) \mid x \in X\}$,

- the composition of $R \subseteq X \times Y$ and $R' \subseteq Y \times Z$ is given by

$$R; R' := \{(x, z) \in X \times Z \mid \exists y \in Y, x \, R \, y \text{ and } y \, R' \, z\}$$

Definition 3.5.2 (Category of relations, indexed point of view)**.** The morphisms in Rel have the following equivalent presentation:

- a morphism from X to Y is given by a relation $r : X \times Y \to \mathrm{Bool}$,

- the identity is given by

$$\mathrm{id}_X : X \times X \to \mathrm{Bool}$$
$$(x, y) \mapsto \begin{cases} 1 \text{ if x = y} \\ 0 \text{ otherwise} \end{cases}$$

- composition of $r : X \times Y \to \mathrm{Bool}$ and $r' : Y \times Z \to \mathrm{Bool}$ is given by

$$(r; r')(x, z) := \bigvee_{y \in Y} r(x, y) \wedge r'(y, z) \tag{3.1}$$

Remark 3.5.3*.* Equation (3.1) is reminiscent of the formula for matrix multiplication $(M \cdot M')_{i,k} = \sum_j M_{i,j} \times M_{j,k}$. Indeed we will see in section 3.8 that just as matrices are presentations for linear maps between vector spaces, relations are also a presentation for another kind of "linear map" between algebraic structures called *suplattices*.

The following few facts explain how to endow Rel with the structure of a Seely category.

Fact 3.5.4. *The category* Rel *admits both finite products and coproducts, and they coincide (i.e.* Rel *admits biproducts). The biproducts are given by underlying coproducts in* Set*. For instance* $X \times_{\mathrm{Rel}} Y = X \sqcup_{\mathrm{Rel}} Y = X \sqcup_{\mathrm{Set}} Y$. *The terminal (and initial) element in* Rel *is the empty set* $\emptyset$.

Fact 3.5.5. *The category* Rel *admits a symmetric monoidal closed structure given by the cartesian product of underlying sets (and with unit a singleton set* 1*), i.e.*

$$X \otimes_{\mathrm{Rel}} Y := X \times_{\mathrm{Set}} Y$$
$$\mathbf{1}_{\mathrm{Rel}} := 1_{\mathrm{Set}}$$

The monoidal closure is given by

$$X \multimap_{\mathrm{Rel}} Y \quad := \quad X \times_{\mathrm{Set}} Y,$$

by virtue of the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Rel}} \left(X \otimes_{\mathrm{Rel}} Y, Z \right) &\simeq P((X \times_{\mathrm{Set}} Y) \times_{\mathrm{Set}} Z) \\
&\simeq P(X \times_{\mathrm{Set}} (Y \times_{\mathrm{Set}} Z)) \\
&\simeq \mathrm{Hom}_{\mathrm{Rel}} \left(X, Y \otimes_{\mathrm{Rel}} Z \right)
\end{aligned}$$

natural in X, Y, Z as objects of Rel.

Remark 3.5.6. The fact those last isomorphisms are natural in X, Y, Z as objects of Rel does not follow tautologically from their naturality in X, Y, Z as objects of Set.

Fact 3.5.7. *From the definition of the internal hom in* Rel *it follows easily that* Rel *is moreover $*$-autonomous, with dualizing object given by any singleton set $\{\star\}$.*

We now describe the multiset comonad on Rel, obtained by lifting the multiset monad on Set. While the usual definition of relations is fibred (i.e. $R \subseteq X \times Y$), the usual definition for multisets is indexed, so we recall that one first.

Definition 3.5.8 (*Multisets, indexed point of view*)**.** Given a set X, a *multiset* on X is a map $\mathbf{m} : X \to \mathbb{N}$ with finite support, i.e. such that the set

$$\{x \in X \mid \mathbf{m}(x) \neq 0\}$$

is finite. The set of multisets on X is written $\mathrm{Mul}\,(X)$.

The construction $X \mapsto \mathrm{Mul}\,(X)$ determines a functor $\mathrm{Mul} : \mathrm{Set} \to \mathrm{Set}$ acting on $f : X \to Y$ by

$$\mathrm{Mul}\,(X) \to \mathrm{Mul}\,(Y)$$

$$(\mathbf{m} : X \to \mathbb{N}) \mapsto \left(y \mapsto \sum_{x \in f^{-1}(y)} \mathbf{m}(x) \right)$$

Intuitively, a multiset $\mathbf{m}$ on X is a finite collection of elements of X that are allowed to appear more than once. The number $\mathbf{m}(x) \in \mathbb{N}$ denotes the "number of occurrences" of the element x in the multiset $\mathbf{m}$. In the fibred point of view, we take this intuition of "finite collection" more seriously by saying a multiset *is* a finite set whose elements are "colored" by elements of X.

Definition 3.5.9 (*Multisets, fibred point of view*)**.** Given a set X, a multiset on X is a pair (m, f) where $n \in \mathbb{N}$ is a natural number and f is map $f : \{1, \dots, n\} \to X$. Multisets are considered up to the equivalence relation given by $(m, f) \sim (n, g)$ if $m = n$ and there is a permutation $\sigma \in \mathfrak{S}_n$ such that $f = g \circ \sigma$.

The functoriality of Mul is given on $f : X \to Y$ by

$$\mathrm{Mul}\,(X) \to \mathrm{Mul}\,(Y)$$

$$(m, g) \mapsto (m, f \circ g)$$

Fact 3.5.10. *The above two definitions yield naturally isomorphic functors* Set $\to$ Set. *The natural isomorphism acts on X by mapping a pair (m, f) with $m \in \mathbb{N}$ and $f : \{1, \dots, m\} \to X$ to the map*

$$X \to \mathbb{N}$$

$$x \mapsto \# f^{-1}(x)$$

Notation 3.5.11 (List notation for multisets)**.** Given a set X and a tuple $(x_1, \ldots, x_n)$ of elements of X, we write $[x_1; \ldots; x_n]$ for the multiset on X given in the fibred point of view as $(n, x_{(-)})$. Every element of $\mathrm{Mul}\,(X)$ can be written in such a way, although in a very non-canonical manner: multisets correspond precisely to lists up to permutation.

The monad laws are given as follows.

Definition 3.5.12 (*Multiset monad laws, indexed point of view*)**.** The unit map $\eta_X : X \to \mathrm{Mul}\,(X)$ is given by

$$\eta_X(x)(x') := \delta_x^{x'}$$

i.e. $\eta_X(x)(x') = 1$ when $x = x'$ and 0 otherwise.

The multiplication map $\mu_X : \mathrm{Mul}\,(\mathrm{Mul}\,(X)) \to \mathrm{Mul}\,(X)$ is given by

$$\mu_X(\mathbf{m})(x) := \sum_{\mathbf{n} \in \mathrm{Mul}\,(X)} \mathbf{m}(\mathbf{n}) \times \mathbf{n}(x)$$

Definition 3.5.13 (*Multiset monad laws, fibred point of view*)**.** The unit map $\eta_X : X \to \mathrm{Mul}\,(X)$ is given by

$$\eta_X(x) := (1, \overline{x})$$

where $\overline{x}$ is the unique map $\{1\} \to X$ that maps 1 to x.

The multiplication map $\mu_X : \mathrm{Mul}\,(\mathrm{Mul}\,(X)) \to \mathrm{Mul}\,(X)$ is given by

$$\mu_X((m, (n_i, f_i)_{1 \le i \le m})) \mapsto \Big(\sum_{i=1}^m n_i, [g_i]_{1 \le i \le n} \Big)$$

where the notation $[g_i]_{1 \le i \le n}$ is used after composition with the isomorphism

$$\{1, \ldots, n_1 + \cdots + n_m\} \simeq \{1, \ldots, n_1\} \sqcup \cdots \sqcup \{1, \ldots, n_m\}.$$

The following results are very much standard, but we will prove them using a novel approach in section 3.6.2:

Fact 3.5.14. *The multiset monad on* Set *lifts to a lax monoidal comonad* Mul *on* Rel *.*

Fact 3.5.15. *The category* Rel *endowed with its symmetric monoidal closed structure* $(\otimes, \mathbf{1}, \multimap)$*, its cartesian structure* $(\sqcup_{\mathrm{Set}}, 1_{\mathrm{Set}})$ *and its comonad* Mul *forms a Seely category, hence a model of intuitionistic linear logic.*

Using fact 3.4.12, we immediately deduce

Corollary 3.5.16. *There is a linear/non-linear adjunction*

$$(\mathrm{Rel}_{\mathrm{Mul}}, \&) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (\mathrm{Rel}, \otimes)$$

between Rel *and the Kleisli category for the comonad* Mul *on* Rel *.*

Remark 3.5.17*.* It turns out that the multiset comonad on Rel is actually a free exponential, in the sense that $\mathrm{Mul}\,(X)$ is the cofree commutative comonoid on X in Rel, so that $(\mathrm{Rel}, \mathrm{Mul})$ is not only a Seely category but also a Lafont category.

A way less-known fact is the following.

Proposition 3.5.18. *There is a linear/non-linear adjunction*

$$(\mathrm{Set}, \times) \; \underset{P}{\overset{\perp}{\underset{\longleftarrow}{\longrightarrow}}} \; (\mathrm{Rel}, \otimes)$$

and hence by facts 3.4.11 and 3.4.14 a Seely category structure on Rel *with the powerset comonad on* Rel.

Proof. The category Rel can be seen as the Kleisli category of the powerset monad P on Set, hence there is such a free/forgetful adjunction. The fact that the left adjoint is symmetric monoidal comes from the fact that the tensor product in Rel is defined as the cartesian product of underlying sets. □

Remark 3.5.19. In section 8.7 of his survey [Mel09], Paul-André Melliès explains why there is no way to equip Rel with a Seely structure whose underlying comonad is given by P. His argument is that the dereliction relation $\mathsf{d}_X : PX \nrightarrow X$ should be given by $\{(\{x\}, x) \mid x \in X\}$ and that such a d_X is not natural in X. However, unfolding the dereliction coming from the previous linear/non-linear adjunction, we see that it should instead be

$$\mathsf{d}_X : PX \nrightarrow X$$
$$\{(U, x) \mid x \in U \subseteq X\}.$$

He also remarks that a variant by Glynn Winskel circumvents this by considering posets instead of sets [Win99]. However as we have just seen there is no need to resort to posets for the powerset comonad to define an exponential on Rel.

In the rest of this chapter, we will explore other points of views on the relational model and its variants. Notably in sections 3.8 and 3.9 we will see when we actually need to generalize from sets to posets. But first, we take a detour to explore abstract properties that make the Seely structure on Rel work.

3.6 ⋆ An abstract interlude on categories of relations

The goal of this section is to give an abstract account of the Seely category structure on Rel. This structure on Rel is obtained by lifting structures already existing on the category Set, e.g. the tensor product $\otimes$ on Rel coming from the cartesian product $\times$ of sets.

The idea is that the operation of constructing the category Rel from the category Set can be generalized to a construction of categories of internal relations $\mathcal{C} \mapsto \mathrm{Rel}\,(\mathcal{C})$, and this construction exhibits a functoriality in a 2-categorical sense (theorem 3.6.27). However the construction of Rel $(\mathcal{C})$ does not make sense for an arbitrary category $\mathcal{C}$. In general, categories of internal relations have been studied in categories with pullbacks equipped with a suitable factorization system (see [Kaw73] for the first general construction and Milius' thesis [Mil00] for a more modern treatment), but to the author's knowledge there is no account of the functoriality of the construction $\mathcal{C} \mapsto \mathrm{Rel}\,(\mathcal{C})$ in the literature. For our purposes we chose to work in the lesser generality of regular categories (definition 3.6.5). A good survey of that notion can be found in chapter 2 of [Bor94b], although we ask for more axioms on our regular categories here to be able to state functoriality of the Rel $(-)$ construction.

In section 5.6.2, we will carry out similar reasonings for (wild) categories of spans in the setting of homotopy type theory. As we will see, the results in that setting will be significantly easier to state and prove, as they will only rely on some existence and preservation of pullbacks, and not on the additional structures required in regular categories. In that sense, categories

of spans are "simpler" than categories of relations, even though they require working higher-categorically to make full sense of. Because of that, this section may be safely skipped on a first reading and read only after section 5.6.2 to see what additional technicalities emerge when considering relations instead of general spans.

3.6.1 Internal relations and their functoriality

Definition 3.6.1. Let $\mathcal{C}$ be a category. A morphism $f : x \to y$ in $\mathcal{C}$ is called a *regular epimorphism* if it is a coequalizer of some pair of maps $g, g' : z \to x$ in $\mathcal{C}$:

$$z \underset{g'}{\overset{g}{\rightrightarrows}} x \xrightarrow{f} y$$

When $\mathcal{C}$ admits pullbacks, there is a canonical choice for such a pair:

Definition 3.6.2. Let $f : x \to y$ be a morphism in $\mathcal{C}$. The *kernel pair* of f is the pair of projections from its pullback with itself:

$$x \times_y x \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} x \xrightarrow{f} y$$

Fact 3.6.3 ([Bor94a, Prop 2.5.7-8])**.** *A morphism with a kernel pair is a regular epimorphism if and only if it is the coequalizer of its kernel pair.*
Reciprocally, if a kernel pair has a coequalizer, then it is the kernel pair of its coequalizer.

Remark 3.6.4. Every regular epimorphism in a category is in particular an epimorphism in the usual sense. Beware that this is no longer true in the setting of higher categories.

To exhibit the 2-functoriality of the Rel construction, we use a slightly less general definition of regular category than in [Bor94b, Def 2.1.1].

Definition 3.6.5. A *regular category* is a category with pullbacks and binary products such that

1. every kernel pair has a coequalizer,

2. regular epimorphisms are preserved under pullback, i.e. for every pullback square

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ f' \downarrow & \lrcorner & \downarrow f \\ \bullet & \longrightarrow & \bullet \end{array}$$

with f a regular epimorphism, then f' must also be a regular epimorphism.

Example 3.6.6. The category Set is a regular category. Every abelian category is regular. Every topos is a regular category.

In a regular category, monomorphisms will be denoted $x \hookrightarrow y$ and regular epimorphisms $x \twoheadrightarrow y$.

Fact 3.6.7 ([Bor94b, Thm 2.1.3, Prop 2.1.4])**.** *In a regular category $\mathcal{C}$, the monomorphisms and the regular epimorphisms form an* orthogonal factorization system*, i.e. every morphisms $f : x \to y$ factors as $f = m \circ e$*

$$x \underset{e}{\twoheadrightarrow} z \underset{m}{\overset{f}{\hookrightarrow}} y$$

and for every commutative diagram

$$
\begin{array}{ccc}
x & \xrightarrow{\ f\ } & y \\
{\scriptstyle e}\downarrow & {\scriptstyle \exists! h}\nearrow & \downarrow{\scriptstyle m} \\
z & \xrightarrow{\ g\ } & t
\end{array}
$$

there exists a unique $h : z \to y$ that makes the two triangles commute.

The general theory of orthogonal factorization systems implies in particular that

1. the factorization of f as $m \circ e$ is unique up to unique isomorphism,

2. that regular epimorphisms are stable under composition,

3. and that a morphism is both a monomorphism and a regular epimorphism if and only if it is an isomorphism.

Equipped with the notion of regular category, we can now define general categories of relations.

Definition 3.6.8. Let $\mathcal{C}$ be a regular category. The *category of relations in $\mathcal{C}$* or *category of relations internal to $\mathcal{C}$*, denoted $\mathrm{Rel}\,(\mathcal{C})$ is defined as follows:

1. its objects are the objects of $\mathcal{C}$,

2. morphisms $r : x \nrightarrow y$ are monomorphisms $r : u \hookrightarrow x \times y$ (considered up to isomorphism in $\mathcal{C}_{/x\times y}$)

3. the identity morphism is given by the diagonal map

$$
\Delta_x : x \xrightarrow{\langle \mathrm{id}_x, \mathrm{id}_x \rangle} x \times x,
$$

4. the composition of $r; r'$ of $r : u \hookrightarrow x \times y$ and $r' : v \hookrightarrow y \times z$ is given by the factorization

$$
u \times_y v \xrightarrow{\quad\quad} w \overset{\langle \pi_1 \circ r \circ \pi_1, \pi_2 \circ r' \circ \pi_2 \rangle}{\underset{r;r'}{\hookrightarrow}} x \times z
$$

Example 3.6.9. In the case $\mathcal{C} = \mathrm{Set}$, this definition recovers the usual category $\mathrm{Rel} = \mathrm{Rel}\,(\mathrm{Set})$ of sets and relations.

Remark 3.6.10. An internal relation $r : u \hookrightarrow x \times y$ can equivalently be described as a pair $(r_1 : u \to x, r_2 : u \to y)$ such that $(r_1, r_2) : u \to x \times y$ is a monomorphism. This is equivalent to the condition that r_1 and r_2 are *jointly monomorphic*, in the sense that whenever $r_1 \circ f = r_1 \circ g$ and $r_2 \circ f = r_2 \circ g$, then $f = g$.

Fact 3.6.11. *Let $\mathcal{C}$ be a regular category. Every morphism $f : x \to y$ in $\mathcal{C}$ defines a relation $\lambda f : x \nrightarrow y$ defined by*

$$
\lambda f : x \xrightarrow{\langle \mathrm{id}_x, f \rangle} x \times y.
$$

Moreover, λ is compatible with composition:

$$
\lambda(g \circ_{\mathcal{C}} f) = \lambda g \circ_{\mathrm{Rel}\,(\mathcal{C})} \lambda f
$$

so that we get a functor

$$\lambda : \mathcal{C} \to \mathrm{Rel}\,(\mathcal{C})$$

acting as the identity on objects.

Proof. The morphism $\lambda f := \langle \mathrm{id}_x, f \rangle : x \to x \times y$ is a monomorphism since it admits a left inverse given by $\pi_1 : x \times y \to x$, so it is indeed a relation from x to y. The compatibility with composition $\lambda(g \circ f) = \lambda g \circ \lambda f$ follows from the fact that the middle square in the following diagram is a pullback square:

$\square$

Fact 3.6.12. *There is an involutive equivalence of categories $(-)^* : \mathrm{Rel}\,(\mathcal{C})^{\mathrm{op}} \to \mathrm{Rel}\,(\mathcal{C})$ that acts as the identity on objects and maps any relation $r : u \hookrightarrow x \times y$ to its composition with the symmetry automorphism $x \times y \xrightarrow{\simeq} y \times x$.*

Remark 3.6.13. In $\mathrm{Rel}\,(\mathcal{C})$, the hom-sets can be endowed with the partial order given by inclusion of relations, and composition is a monotone function of these orderings. This means $\mathrm{Rel}\,(\mathcal{C})$ can be seen as canonically enriched over partial orders, or in other words it is a $(1, 2)$-category. In that context, a relation $r : x \nrightarrow y$ is a left adjoint (in the sense of a left adjoint morphism in a bicategory, defined in terms of a unit and a counit) if and only if it is of the form λf for some $f : x \to y$, and its right adjoint is $(\lambda f)^*$ [Bor94b, Thm 2.8.4].

We now turn to the functoriality of the construction $\mathcal{C} \mapsto \mathrm{Rel}\,(\mathcal{C})$. For a functor $F : \mathcal{C} \to \mathcal{D}$ between regular categories to lift to a functor between their categories of relations, it must preserve some of the structure of these categories. A reasonable candidate would be that F must preserve pullbacks and regular epimorphisms, however our main interest is to show that the multiset functor $\mathrm{Mul} : \mathrm{Set} \to \mathrm{Set}$ lifts to a functor $\mathrm{Rel} \to \mathrm{Rel}$, and this functor does not satisfy these hypotheses (see remark 3.6.34). Hence we use the following weaker and less-conventional definition.

Definition 3.6.14. A functor $F : \mathcal{C} \to \mathcal{D}$ between regular categories is said to be *relational* if

1. it preserves regular epimorphisms,

2. for every pullback square

$$
\begin{array}{ccc}
w & \xrightarrow{k} & y \\
{\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
x & \xrightarrow{f} & z
\end{array}
$$

in $\mathcal{C}$, the induced map

$$\langle Fh, Fk \rangle : F(w) \to F(x) \times_{F(z)} F(y)$$

is a regular epimorphism.

It is said to be *regular* if the following condition applies instead of condition 2:

2′. *F* preserves pullbacks.

Proposition 3.6.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between regular categories. If F is a regular functor, then it is a relational functor.*

Proof. Indeed, *F* preserving pullbacks precisely means that the canonical maps

$$\langle Fh, Fk \rangle : F(w) \to F(x) \times_{F(z)} F(y)$$

as above are isomorphisms, and hence a fortiori regular epimorphisms. □

Remark 3.6.16. The reader may be familiar with regular categories in the context of *regular logic*, where one includes the existence of all finite limits in their definition, and regular functors are assumed to preserve all finite limits [But98]. The "regular functors" we define here are weaker only in that they do not preserve terminal objects (as a functor that preserves pullbacks and terminal objects preserves all finite limits). Our relational functors however are way more general, as they need not preserve any limit whatsoever.

Proposition 3.6.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be a relational functor between regular categories. Then F preserves monomorphisms. Since it also preserves regular epimorphisms, it preserves the regular epi/mono factorization of morphisms. More explicitly, if $f = m \circ e$ is a regular epi/mono factorization of f in $\mathcal{C}$, then $F(f) = F(m) \circ F(e)$ is a regular epi/mono factorization of $F(f)$ in $\mathcal{D}$.*

Proof. A morphism $m : x \to y$ in a category is a monomorphism if and only if the commutative square

$$
\begin{array}{ccc}
x & \xrightarrow{\mathrm{id}_x} & x \\
{\scriptstyle \mathrm{id}_x}\downarrow & & \downarrow{\scriptstyle m} \\
x & \xrightarrow{m} & y
\end{array}
$$

is a pullback square. Let $m : x \hookrightarrow y$ be a monomorphism in $\mathcal{C}$. Since *F* is a relational functor, the diagonal map Δ in the following diagram is a regular epimorphism.

Moreover, since $\pi_1 \circ \Delta = \mathrm{id}_{F(x)}$, Δ is a (split) monomorphism. By virtue of being both a regular epimorphism and a monomorphism, Δ is an isomorphism (by fact 3.6.7), so the following square

$$
\begin{array}{ccc}
F(x) & \xrightarrow{\mathrm{id}} & F(x) \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle Fm} \\
F(x) & \xrightarrow{Fm} & F(y)
\end{array}
$$

is a pullback square, which entails that *Fm* is a monomorphism. □

To prove the next proposition, we first need a lemma about an additional compatibility between regular epimorphisms and pullbacks in a regular category.

Lemma 3.6.18. *Let*

be a diagram in a regular category. Then the canonical map

$$\langle f, g \rangle : u \times_x v \to u' \times_x v'$$

is a regular epimorphism.

Proof. We have the following three pullback squares.

Since f is a regular epimorphism in a regular category, $f \times_x v'$ must also be a regular epi, and so must $f \times_x v$. Symmetrically, $u' \times_x g$ and $u \times_x g$ must be regular epimorphisms too. Since regular epimorphisms are stable under composition, the map

$$\langle f, g \rangle = (u' \times_x g) \circ (f \times_x v) = (f \times_x v') \circ (u \times_x g)$$

is also regular epimorphism. $\qquad\square$

Proposition 3.6.19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a relational functor between regular categories. Then it induces a functor $\mathrm{Rel}\,(F) : \mathrm{Rel}\,(\mathcal{C}) \to \mathrm{Rel}\,(\mathcal{D})$ such that:*

1. *$\mathrm{Rel}\,(F)$ acts as F on objects,*

2. *$\mathrm{Rel}\,(F)$ maps a relation $r : u \hookrightarrow x \times y$ in $\mathcal{C}$ to the regular epi/mono factorization of the composite*

$$F(u) \xrightarrow{F(r)} F(x \times y) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} F(x) \times F(y) \tag{3.2}$$

Proof. To show $\mathrm{Rel}\,(F)$ defines a functor, we must show it preserves identities and composition of morphisms.

Identities. Let $x \in \mathcal{C}$, its identity relation is given by the diagonal map $\Delta_x : x \to x \times x$, characterized as the unique morphism $x \to x \times x$ whose composition with both projections gives back the identity morphism of x in $\mathcal{C}$, as shown on the left below.
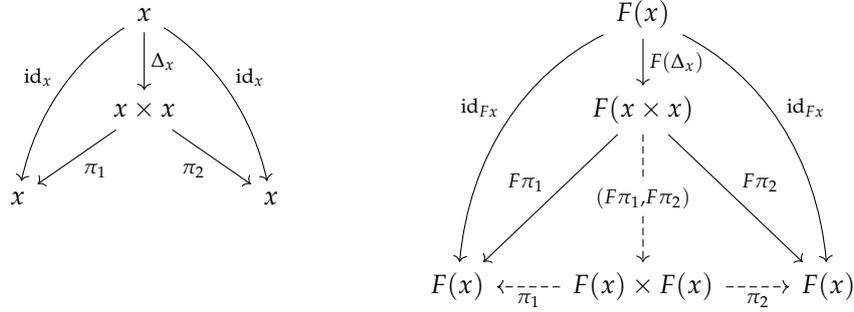
By functoriality of F, this diagram is mapped to the one in solid arrows on the right. The rest of the diagram on the right commutes by the universal property of the product. Hence $(F\pi_1, F\pi_2) \circ F(\Delta_x) \circ \pi_i = \mathrm{id}_{Fx}$ for $i = 1, 2$, so $(F\pi_1, F\pi_2) \circ F(\Delta_x) = \Delta_{F(x)}$. Since $\Delta_{F(x)}$ is a monomorphism, factoring it as a regular epi/mono in $\mathcal{D}$ yields itself, hence $\mathrm{Rel}\,(F)(\mathrm{id}_x) = \mathrm{id}_{\mathrm{Rel}\,(F)(x)}$ in $\mathrm{Rel}\,(\mathcal{D})$.

Composition. Let $r : u \hookrightarrow x \times y$ and $r' : v \hookrightarrow y \times z$ be relations in $\mathcal{C}$. Their composition is given by the factorization

$$u \times_y v \xrightarrow{\quad\longtwoheadrightarrow\quad} w \xhookrightarrow[r;r']{\langle \pi_1 \circ r \circ \pi_1, \pi_2 \circ r' \circ \pi_2 \rangle} x \times z$$

The relations r and r' are respectively mapped by $\mathrm{Rel}\,(F)$ to the relations given by the factorizations

We have the following commutative diagram.

The map $F(u \times_y v) \to F(u) \times_{F(y)} F(v)$ is a regular epi by assumption that F is a relational functor. The map $F(u) \times_{F(y)} F(v) \to u' \times_{F(y)} v'$ is a regular epi by lemma 3.6.18. By uniqueness of regular

epi/mono factorizations, there must be an isomorphism $\theta : w' \to t$ making the following triangle commute.

$$
\begin{array}{ccc}
w' & \xdashrightarrow{\quad\theta\quad} & t \\
& \searrow \qquad\qquad \swarrow & \\
\mathrm{Rel}\,(F)(r;r') & & \mathrm{Rel}\,(F)(r);\mathrm{Rel}\,(F)(r') \\
& F(x) \times F(z) &
\end{array}
$$

Hence $\mathrm{Rel}\,(F)(r;r') = \mathrm{Rel}\,(F)(r);\mathrm{Rel}\,(F)(r')$ as relations $F(x) \nrightarrow F(z)$. $\qquad\square$

Finally we turn to natural transformations.

Definition 3.6.20. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between categories, and $\alpha : F \Rightarrow G$ a natural transformation. We say α is *cartesian* if for every morphism $f : x \to y$ in $\mathcal{C}$, the naturality square

$$
\begin{array}{ccc}
Fx & \xrightarrow{\ Ff\ } & Fy \\
\alpha_x \downarrow & & \downarrow \alpha_y \\
Gx & \xrightarrow[Gf]{} & Gy
\end{array}
$$

is a pullback square.

Cartesian natural transformations are a well-studied notion in category theory (notably in the study of polynomial functors [GK13]). They are a natural notion of 2-morphism between regular functors (and more generally pullback-preserving functors), and they will be very useful in our proof of the functoriality of the span construction in section 5.6. However in our study of internal relations, just like we needed to weaken regular functors to instead consider relational functors, we're going to need a weaker assumption than cartesianness on natural transformations.

Definition 3.6.21. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between categories, and $\alpha : F \Rightarrow G$ a natural transformation. We say α is *relational* if, for every map $f : x \to y$ in $\mathcal{C}$, the canonical map

$$
Fx \to Gx \times_{Gy} Fy
$$

induced by the naturality of α is a regular epimorphism in $\mathcal{D}$.

Remark 3.6.22. A natural transformation is cartesian if and only if the above canonical map is an isomorphism. Since every isomorphism is a regular epimorphism, it follows that every cartesian natural transformation is a relational natural transformation.

Proposition 3.6.23. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be relational functors between regular categories, and let $\alpha : F \Rightarrow G$ be a relational natural transformation. Then α lifts to a natural transformation*

$$
\mathrm{Rel}\,(\alpha) : \mathrm{Rel}\,(F) \Rightarrow \mathrm{Rel}\,(G) : \mathrm{Rel}\,(\mathcal{C}) \to \mathrm{Rel}\,(\mathcal{D})
$$

given on objects by $\mathrm{Rel}\,(\alpha)_x = \lambda(\alpha_x)$.

Proof. Let $r : u \hookrightarrow x \times y$ be a relation in $\mathcal{C}$. We need to show the following square in $\mathrm{Rel}\,(\mathcal{D})$ commutes.

$$
\begin{array}{ccc}
Fx & \xmapsto{\ \mathrm{Rel}\,(F)(f)\ } & Fy \\
\lambda(\alpha_x) \downarrow & & \downarrow \lambda(\alpha_y) \\
Gx & \xmapsto[\mathrm{Rel}\,(G)(f)]{} & Gy
\end{array}
$$

Write $r = \langle r_1, r_2 \rangle$ with $r_1 : u \to x$, $r_2 : u \to y$ so that we have the following diagram of spans.

$$
\begin{array}{ccccc}
Fx & \xleftarrow{\;Fr_1\;} & Fu & \xrightarrow{\;Fr_2\;} & Fy \\
{\scriptstyle \mathrm{id}_{Fx}}\big\uparrow & & & & \big\uparrow{\scriptstyle \mathrm{id}_{Fy}} \\
Fx & & & & Fy \\
{\scriptstyle \alpha_x}\big\downarrow & & & & \big\downarrow{\scriptstyle \alpha_y} \\
Gx & \xleftarrow{\;Gr_1\;} & Gu & \xrightarrow{\;Gr_2\;} & Gy
\end{array}
$$

To compose relations, we compose the spans by pullback and then factorize as regular epi/mono the canonical maps from the pullbacks to $Fx \times Gy$. Since pulling back along an identity morphism yields and identity morphism, we can compose spans in the following way

$$
\begin{array}{ccccccc}
Fx & \xleftarrow{\qquad Fr_1 \qquad} & & Fu & \xrightarrow{\;Fr_2\;} & Fy \\
{\scriptstyle \mathrm{id}_{Fx}}\big\uparrow & & & \big\uparrow {\scriptstyle \mathrm{id}_{Fu}} & & \big\uparrow {\scriptstyle \mathrm{id}_{Fy}} \\
Fx & \xleftarrow{\;\pi_1\;} & Fx \times_{Gx} Gu & \xleftarrow{\;\;h\;\;} Fu & \xrightarrow{\;Fr_2\;} & Fy \\
{\scriptstyle \alpha_x}\big\downarrow & & \big\downarrow {\scriptstyle \pi_2} & & & \big\downarrow {\scriptstyle \alpha_y} \\
Gx & \xleftarrow{\;Gr_1\;} & Gu & \xrightarrow{\qquad Gr_2 \qquad} & & Gy
\end{array}
$$

and we get a map $h : Fu \to Fx \times_{Gx} Gu$ by naturality of α.

$$
\begin{array}{ccc}
Fu & & \\
\;\;\big\downarrow{\scriptstyle \alpha_u} \;\; \searrow{\scriptstyle h} & \searrow^{Fr_1} & \\
 & Fx \times_{Gx} Gu \xrightarrow{\;\pi_1\;} Fx & \\
 & \big\downarrow{\scriptstyle \pi_2} \qquad\;\; \big\downarrow{\scriptstyle \alpha_x} & \\
 & Gu \xrightarrow{\;Gr_1\;} Gx &
\end{array}
$$

The assumption that α is relational applied to the morphism $r_1 : u \to x$ in $\mathcal{C}$ precisely says that h is a regular epimorphism. Hence we get a commutative diagram

$$
\begin{array}{ccc}
Fu & \overset{\langle Fr_1, \alpha_y \circ Fr_2 \rangle}{\twoheadrightarrow w \hookrightarrow} & Fx \times Gy \\
{\scriptstyle h}\big\downarrow & \big\downarrow{\scriptstyle \simeq} & \big\| \\
Fx \times_{Gx} Gu & \underset{\mathrm{id}_{Fx} \times_{Gx} Gr_2}{\twoheadrightarrow w' \hookrightarrow} & Fx \times Gy
\end{array}
$$

implying the existence of a dotted isomorphism as witnessing the desired equality between compositions of relations. $\qquad\square$

To finally state the functoriality of the construction $\mathcal{C} \mapsto \mathrm{Rel}\,(\mathcal{C})$, we need to show that regular categories, relational functors and relational natural transformations form a 2-category (or more explicitly a sub-2-category of Cat). We first prove the analogous result for regular functors and

cartesian natural transformations, as the proof is significantly shorter and still instructive: in the relational case, the proof has the same core ideas, with the added technical difficulty of checking that everything is compatible with regular epimorphisms instead of isomorphisms.

Proposition 3.6.24. *Restricting the 2-category* Cat *of categories, functors and natural transformations to regular categories, regular functors and cartesian natural transformations yields a sub-2-category* RegCat *.*

Proof. The identity functor $\mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ of a regular category $\mathcal{C}$ is straightforwardly regular (and more generally every equivalence is a regular functor). Regular functors compose because they are defined by preservation of some limits and some class of morphisms.

Identity natural transformations are straightforwardly cartesian (and so are every invertible natural transformations). Vertical composition of natural transformations preserves cartesianness because the juxtaposition of pullback squares is again a pullback square. We are left with checking that horizontal composition also preserves cartesianness. We do that by showing that it is preserved by left and right whiskering.

Consider the following diagram of regular categories, regular functors and cartesian natural transformations in Cat .

$$\mathcal{B} \xrightarrow{\;K\;} \mathcal{C} \overset{F}{\underset{G}{\Longrightarrow}}{\Downarrow \alpha} \mathcal{D} \xrightarrow{\;H\;} \mathcal{E}$$

The left whiskering $H * \alpha : H \circ F \Rightarrow H \circ G$ is defined by $(H * \alpha)_x := H(\alpha_x)$. Since H is regular, it preserves pullbacks, so the naturality squares of $H * \alpha$ are pullbacks.

The right whiskering $\alpha * K : F \circ K \Rightarrow G \circ K$ is defined by $(\alpha * K)_x := \alpha_{K(x)}$. For any $f : x \to y$ in $\mathcal{B}$, the naturality square of $\alpha * K$ for f is just the naturality square of α for Kf, which is a pullback square by assumption on α.

Since horizontal composition of natural transformations can be defined by vertical composition and whiskering, we get that cartesianness is preserved by horizontal composition. Hence RegCat defines a sub-2-category of Cat . $\qquad\square$

Proposition 3.6.25. *Restricting the 2-category* Cat *to regular categories, relational functors and relational natural transformations yields a sub-2-category* RelCat *.*

Proof. As before, we need to show that RelCat is closed under identities, composition of functors, vertical composition of natural transformations, and left and right whiskering.

Identities. As noted in the proof of proposition 3.6.24, identity functors (more generally, equivalences of categories) are regular and hence relational by proposition 3.6.15. Identity natural transformations (and more generally, all natural isomorphisms) are cartesian and hence also relational by remark 3.6.22.

Composition of functors. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be relational functors between regular categories. The functor $G \circ F : \mathcal{C} \to \mathcal{E}$ preserves regular epimorphisms since F and G both do. Let

$$\begin{array}{ccc} w & \xrightarrow{\;k\;} & y \\ {\scriptstyle h}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ x & \xrightarrow[\;f\;]{} & z \end{array}$$

be a pullback square in $\mathcal{C}$. We have the following commutative diagram

$$\begin{array}{c}
GFw \\
\quad\searrow^{G(\langle F\pi 1, F\pi_2\rangle)} \qquad\qquad \searrow^{GFk} \\
\quad G(Fx \times_{Fz} Fy) \\
\qquad\qquad \searrow^{\langle G\pi_1, G\pi_2\rangle} \\
\searrow^{GFh} \qquad\qquad GFx \times_{GFz} GFy \xrightarrow{\pi_2} GFy \\
\qquad\qquad\qquad \downarrow^{\pi_1} \quad\lrcorner \qquad\qquad \downarrow^{GFg} \\
\qquad\qquad GFx \xrightarrow{GFf} GFz
\end{array}$$
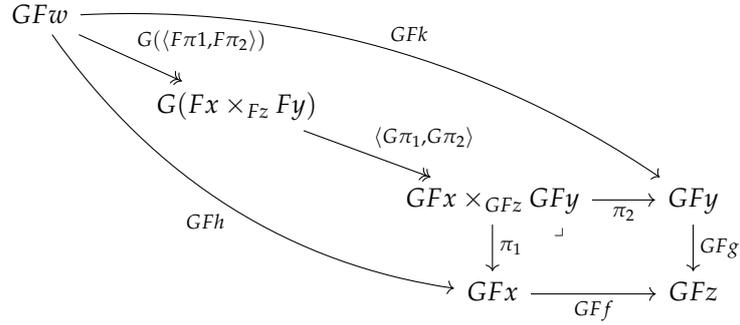
where

- the map
$$\langle G\pi_1, G\pi_2\rangle : G(Fx \times_{Fz} Fy) \to GFx \times_{GFz} GFy$$
  is a regular epi because G is a regular functor,

- the map
$$G(\langle F\pi_1, F\pi_2\rangle) : GFw \to G(Fx \times_{Fz} Fy)$$
  is a regular epi because F is a regular functor and G preserves regular epis.

Since regular epimorphisms are stable under composition, this entails that the canonical map

$$\langle GF\pi_1, GF\pi_2\rangle : GFw \to GFx \times_{GFz} GFy$$

is a regular epi. This concludes the proof that $G \circ F$ is a regular functor.

Vertical composition of natural transformations. Consider the following diagram of regular categories, relational functors and relational natural transformations.

$$\mathcal{C} \quad\begin{array}{c}\xrightarrow{\quad F\quad} \\ \Downarrow\alpha \\ \xrightarrow{\ G\ } \\ \Downarrow\beta \\ \xrightarrow{\quad H\quad}\end{array}\quad \mathcal{D}$$

and let $f : x \to y$ be a morphism in $\mathcal{C}$. We have the following commutative diagram.

$$\begin{array}{c}
Fx \xrightarrow{\qquad\qquad Ff \qquad\qquad} Fy \\
\downarrow^{\alpha_x} \quad\searrow \qquad\qquad \nearrow \qquad \uparrow \\
\quad Gx \times_{Gy} Fy \longrightarrow Hx \times_{Hy} Fy \qquad\qquad\qquad \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow^{\alpha_y} \\
Gx \quad\xrightarrow{Gf}\qquad \lrcorner \qquad \lrcorner \\
\quad\searrow \qquad\qquad Hx \times_{Hy} Gy \longrightarrow Gy \\
\qquad\beta_x \qquad\qquad\qquad \downarrow \quad\lrcorner \qquad \downarrow^{\beta_y} \\
\qquad\qquad Hx \xrightarrow{\ Hf\ } Hy
\end{array}$$
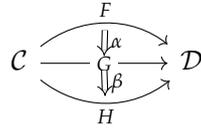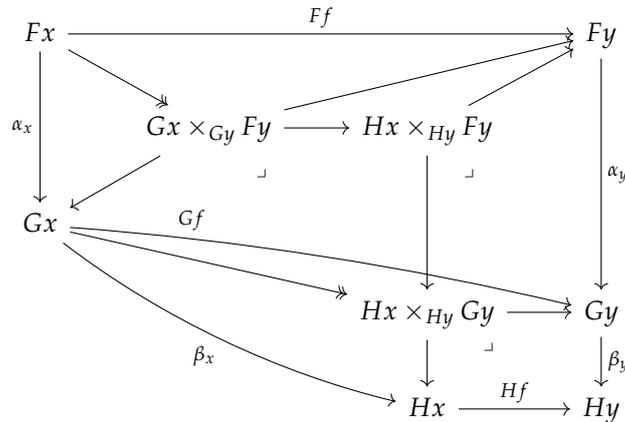
The canonical map
$$Gx \twoheadrightarrow Hx \times_{Hy} Gy$$
is a regular epi by relationality of β. Since regular epis are stable under pullbacks in regular categories, the map
$$Gx \times_{Gy} Fy \to Hx \times_{Hy} Fy$$
is also a regular epi. The canonical map
$$Fx \twoheadrightarrow Gx \times_{Gy} Fy$$
is a regular epi by relationality of α. Hence the composite map
$$Fx \xrightarrow{\hspace{1.5cm}} Gx \times_{Gy} Fy \xrightarrow{\hspace{1cm}} Hx \times_{Hy} Fy$$
$$\overset{\langle \beta_x \circ \alpha_x, Ff \rangle}{}$$
is a regular epi and $\beta \circ \alpha$ is a relational natural transformation.

Whiskering. Consider the following diagram of regular categories, relational functors and relational natural transformations.

$$\mathcal{B} \xrightarrow{K} \mathcal{C} \underset{G}{\overset{F}{\Rightarrow}} \mathcal{D} \xrightarrow{H} \mathcal{E}$$

Left whiskering. As before, we have $(H * \alpha)_x = H(\alpha_x)$. Let $f : x \to y$ be a morphism in $\mathcal{C}$. We have the following commutative diagram

where the map $H\langle \alpha_x, Ff \rangle$ is a regular epi by relationality of α and the fact that H preserves regular epis, and $\langle H\pi_1, H\pi_2 \rangle$ is a regular epi because H is a relational functor. Hence $H * \alpha$ is a relational natural transformation.

Right whiskering. By definition, $(\alpha * K)_x := \alpha_{K(x)}$. For any $f : x \to y$ in $\mathcal{B}$, the naturality square of $\alpha * K$ for f is just the naturality square of α for Kf, so the condition that $\langle \alpha_{Kx}, KFf \rangle$ be a regular epi is readily verified.

Since RelCat is stable under composition of 1-morphisms, vertical composition of 2-morphisms and left and right whiskering, it is a sub-2-category of Cat as desired. $\square$

Corollary 3.6.26. RegCat *is a sub-2-category of* RelCat *(with the same objects).*

Theorem 3.6.27. *The construction $\mathcal{C} \mapsto \mathrm{Rel}(\mathcal{C})$ defines a 2-functor from* RelCat *to* Cat*.*

Proof. The action on objects, morphisms and 2-morphisms is respectively given by definition 3.6.8, proposition 3.6.19 and proposition 3.6.23. We now need to show that the different kinds of identities and compositions in the 2-category RelCat are preserved by the construction $\mathrm{Rel}(-) :$ RelCat $\to$ Cat.

Identity functors. Let $\mathcal{C}$ be a regular category. The functor

$$\mathrm{Rel}(\mathrm{id}_\mathcal{C}) : \mathrm{Rel}(\mathcal{C}) \to \mathrm{Rel}(\mathcal{C})$$

acts as the identity on objects. Given a relation $u \hookrightarrow x \times y$ in $\mathcal{C}$, the diagram (3.2) in the case $F = \mathrm{id}_\mathcal{C}$ becomes

$$u \xrightarrow{\ r\ } x \times y \xrightarrow{\ \mathrm{id}\ } x \times y$$

with u' below and $\mathrm{Rel}(\mathrm{id}_\mathcal{C})(r)$.

We can just take $u' := u$ to witness that $\mathrm{Rel}(\mathrm{id}_\mathcal{C})(r) = r$.

Composition of functors. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ in RelCat. The action of $\mathrm{Rel}(G \circ F)$ on objects is the same as that of $G \circ F$, which is the same as that of $\mathrm{Rel}(G) \circ \mathrm{Rel}(F)$. Now let $r : u \hookrightarrow x \times y$ be a relation in $\mathcal{C}$. Its image by $\mathrm{Rel}(F)$ is given by a factorization

$$F(u) \xrightarrow{\ F(r)\ } F(x \times y) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} F(x) \times F(y)$$

with u' below and $\mathrm{Rel}(F)(r)$.

The following commutative diagram witnesses $\mathrm{Rel}(G)(\mathrm{Rel}(F)(u))$ as a a regular epi/mono factorization of the canonical map $GFu \to GFx \times GFy$.

$$GF(u) \xrightarrow{\ GF(r)\ } GF(x \times y) \xrightarrow{G\langle F\pi_1, F\pi_2 \rangle} G(F(x) \times F(y)) \xrightarrow{\langle G\pi_1, G\pi_2 \rangle} GFx \times GFy$$

with upper arc $\langle GF\pi_1, GF\pi_2 \rangle$ and lower Gu', $G(\mathrm{Rel}(F)(r))$, u'', $\mathrm{Rel}(G)(\mathrm{Rel}(F)(u))$.

The map $GFu \twoheadrightarrow Gu'$ is a regular epi because G preserves regular epis.

Identity natural transformations. Let $F : \mathcal{C} \to \mathcal{D}$ be a relational functor between regular categories. The natural transformation

$$\mathrm{Rel}(\mathrm{id}_F) : \mathrm{Rel}(F) \Rightarrow \mathrm{Rel}(F)$$

is given on an object $x \in \mathrm{Rel}(\mathcal{C})$ by

$$\mathrm{Rel}(\mathrm{id}_F)_x := \lambda(\mathrm{id}_x) : x \xrightarrow{\langle \mathrm{id}_x, \mathrm{id}_x \rangle} x \times x,$$

the diagonal map of x, which is by definition the identity of x in $\mathrm{Rel}(\mathcal{C})$.

Vertical composition of natural transformations. Let
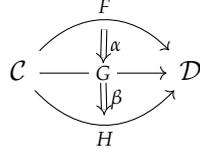
be a diagram in RelCat. We have

$$
\begin{aligned}
\operatorname{Rel}(\beta \circ \alpha)_x &:= \lambda((\beta \circ \alpha)_x) && \text{(by proposition 3.6.23)} \\
&= \lambda(\beta_x \circ \alpha_x) \\
&= \lambda(\beta_x) \circ \lambda(\alpha_x) && \text{(by fact 3.6.11)} \\
&= \operatorname{Rel}(\beta)_x \circ \operatorname{Rel}(\alpha)_x
\end{aligned}
$$

so Rel preserves the vertical composition of 2-morphisms.

Whiskering . Consider the following diagram in RelCat.
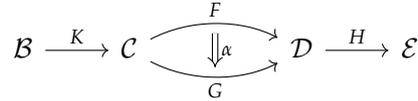
- Left whiskering. Let $x \in \mathcal{C}$. We have

$$
\begin{aligned}
\operatorname{Rel}(H * \alpha)_x &:= \lambda((H * \alpha)_x) \\
&= \lambda(H(\alpha_x)) \\
&= \operatorname{Rel}(H)(\lambda \alpha_x) \\
&= \operatorname{Rel}(H)(\operatorname{Rel}(\alpha)_x) \\
&= (\operatorname{Rel}(H) * \operatorname{Rel}(\alpha))_x.
\end{aligned}
$$

- Right whiskering. Let $x \in \mathcal{B}$. We have

$$
\begin{aligned}
\operatorname{Rel}(\alpha * K)_x &:= \lambda((\alpha * K)_x) \\
&= \lambda(\alpha_{K(x)}) \\
&= \operatorname{Rel}(\alpha)_{K(x)} \\
&= \operatorname{Rel}(\alpha)_{\operatorname{Rel}(K)(x)}
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Not only is the Rel $(-)$ construction functorial, it is also compatible with products of (regular) categories.

Proposition 3.6.28. *The 2-categories* RegCat *and* RelCat *admit finite products and they are computed as in* Cat *.*

Proof. **Nullary product.** The terminal category 1 is regular. Given a regular category $\mathcal{C}$, the unique functor $F : \mathcal{C} \to 1$ preserves all limits since any diagram in 1 is a limit diagram. Moreover,

it maps every morphism to the identity of the unique element of 1, which is a regular epi by virtue of being an isomorphism. In particular F preserves regular epis, and so by it is a regular functor (and a fortiori a regular functor by proposition 3.6.15). Hence 1 is terminal in RegCat (and in RelCat).

Binary products. Let $\mathcal{C}, \mathcal{D}$ be regular categories. Limits and colimits in $\mathcal{C} \times \mathcal{D}$ are computed factor-wise. Hence binary products, pullbacks, monomorphisms, kernel pairs, coequalizers and regular epimorphisms exist in $\mathcal{C} \times \mathcal{D}$ and are preserves by both projections $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$. Hence $\mathcal{C} \times \mathcal{D}$ is a product of $\mathcal{C}$ and $\mathcal{D}$ in RegCat (and in RelCat once again by proposition 3.6.15). $\qquad \square$

Theorem 3.6.29. *The* 2*-functor* Rel : RelCat $\to$ Cat *preserves finite products.*

Proof. There is only one relation in the terminal category 1, hence Rel$(1) \simeq 1$.

Let $\mathcal{C}, \mathcal{D}$ be regular categories. The canonical functor Rel$(\mathcal{C} \times \mathcal{D}) \to$ Rel$(\mathcal{C}) \times$ Rel$(\mathcal{D})$ acts as the identity on objects. It is also fully faithful: since monomorphisms and binary products in $\mathcal{C} \times \mathcal{D}$ are detected factor-wise, any relation $(u, v) \hookrightarrow (c, d) \times (c', d')$ is of the form $(r_\mathcal{C}, r_\mathcal{D})$ with $r_\mathcal{C} : u \hookrightarrow c \times c'$ in $\mathcal{C}$ and $r_\mathcal{D} : v \hookrightarrow d \times d'$ in $\mathcal{D}$, hence

$$\mathrm{Hom}_{\mathrm{Rel}(\mathcal{C} \times \mathcal{D})}((c, d), (c', d')) \simeq \mathrm{Hom}_{\mathrm{Rel}(\mathcal{C}) \times \mathrm{Rel}(\mathcal{D})}((c, d), (c', d')).$$

Thus the functor Rel$(\mathcal{C} \times \mathcal{D}) \to$ Rel$(\mathcal{C}) \times$ Rel$(\mathcal{D})$ is an equivalence, which concludes the proof. $\qquad \square$

Corollary 3.6.30. *The sub-2-category* RegCat *of* RelCat *is stable under cartesian products, so the composite functor* Rel$(-)$: RegCat $\to$ Cat *is cartesian.*

3.6.2 The Seely structure on Rel

Theorems 3.6.27 and 3.6.29 have huge formal consequences. Indeed, 2-functors preserve adjunctions: this means that any adjunction between regular categories consisting of relational functors and relational natural transformations lifts to an adjunction between their categories of relations. Functors between 2-categories also preserve monads, so one can similarly lift "relational monads" to monads on categories of relations. In both cases, since categories of relations are self-dual (fact 3.6.12), this means that we also get opposite adjunctions and comonads for free. Moreover, since Rel$(-)$ preserves cartesian products, it preserves (symmetric) pseudomonoids and morphisms between them. In our case this means the following.

Corollary 3.6.31. *Any symmetric monoidal structure on a regular category $\mathcal{C}$ whose tensor product* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is a relational functor lifts to* Rel$(\mathcal{C})$.

Proof. This is a consequence of theorem 3.6.29 together with the fact that the natural transformations appearing in the definition of (symmetric) monoidal categories are isomorphisms, and hence cartesian. $\qquad \square$

Corollary 3.6.32. *The cartesian symmetric monoidal structure on* Set *lifts to a symmetric monoidal structure on* Rel *as described in fact 3.5.5.*

Proof. The cartesian product functor Set $\times$ Set $\to$ Set preserves pullbacks since limits commute with limits. Every epimorphism in Set is regular, so regular epimorphisms are just surjections, and a cartesian product of surjections is again a surjection. Hence the cartesian product functor Set $\times$ Set $\to$ Set is regular, and a fortiori relational. $\qquad \square$

The general philosophy here is that the heavy abstract work we did in section 3.6 rewards us with automatic constructions of structures on Rel without having to do any computations. Still, this might seem like a lot of work to get something that could have been easily constructed by hand, but there are two non-negligible benefits to this approach.

1. This general framework can give insight into what abstract properties of the category Set make Rel work as a model of linear logic, so that we may easily generalize to models of linear logic in relations internal to other categories,

2. In a higher-categorical setting, constructions relying on explicit computations do not generalize well, due to the need to specify higher coherences. Finding abstract ways to construct familiar structures can thus help construct their analogues in higher categories (see for instance the beginning of section 6.10 on the definition of symmetric monoidal ∞-categories). in chapter 6.

Proposition 3.6.33. *The functor* Mul $: \text{Set} \to \text{Set}$ *is a relational functor.*

Proof. We use the fibred point of view on multisets (definition 3.5.9).

Let $f : X \to Z$ and $g : Y \to Z$ be maps of sets. The canonical map

$$\text{Mul}\,(X \times_Z Y) \to \text{Mul}\,(X) \times_{\text{Mul}\,(Z)} \text{Mul}\,(Y) \tag{3.3}$$

acts as

$$(n, h) \mapsto ((n, \pi_1 \circ h), (n, \pi_2 \circ h)).$$

We show this map is surjective: given $((m, h), (n, k)) \in \text{Mul}\,(X) \times_{\text{Mul}\,(Z)} \text{Mul}\,(Y)$, the pullback condition $\text{Mul}\,(f)(m, h) = \text{Mul}\,(g)(n, k)$ tells us precisely that $m = n$ and $f \circ h = g \circ k$, hence we get a unique map $l : \{1, \dots, n\} \to X \times_Z Y$ making the following diagram commute

The map l gets us an element $(n, l) \in \text{Mul}\,(X \times_Z Y)$ whose image by the map 3.3 is $((m, h), (n, k))$.

It remains to be shown that Mul preserves regular epis, i.e. surjections. Let $f : X \to Y$ be a surjection and $(m, g) \in \text{Mul}\,(Y)$. Surjections in Set are split, i.e. we can find $h : Y \to X$ such that $f \circ h = \text{id}_Y$. Choosing such an h we can build $(m, h \circ g) \in \text{Mul}\,(X)$ and easily verify that

$$\text{Mul}\,(f)(m, h \circ g) := (m, f \circ h \circ g) = (m, g)$$

which shows that $\text{Mul}\,(f)$ is surjective and concludes the proof. $\square$

Remark 3.6.34. The functor Mul $: \text{Set} \to \text{Set}$ does not preserve pullbacks, as can be shown by the following explicit example. Writing 1 for the singleton set $1 = \{0\}$ and Bool $:= \{0, 1\}$, consider the following pullback (actually a product).

We have the following

$$\mathrm{Mul}\,(\mathrm{Bool}\,\times_1 \mathrm{Bool}\,) \to \mathrm{Mul}\,(\mathrm{Bool}\,) \times_{\mathrm{Mul}\,(1)} \mathrm{Mul}\,(\mathrm{Bool}\,)$$
$$[(0,0);(1,1)] \mapsto ([0;1],[0;1])$$
$$[(0,1);(1,0)] \mapsto ([0;1],[0;1])$$

with $[(0,0);(1,1)] \neq [(0,1);(1,0)]$.

Remark 3.6.35. This failure of injectivity of the multiset functor is actually pretty deep: as we will see in section 5.8.3, in the setting of homotopy type theory, the set $\mathrm{Mul}\,(X)$ can be thought of as a *set-truncation* of a more refined *homotopical multiset groupoid* $\mathrm{HMul}\,(X)$, where $\mathrm{HMul}\,:\mathrm{Gpd}\, \to \mathrm{Gpd}\,$ is an actual pullback-preserving pseudo-functor between bicategories.

Remark 3.6.36. Even though the refined homotopical functor $\mathrm{HMul}\,:\mathrm{Gpd}\, \to \mathrm{Gpd}\,$ does preserve pullbacks, it still does not preserve finite products, so it would still not be "regular" in the sense of regular logic (see remark 3.6.16). A high-level explanation for this is the following: we can write

$$\mathrm{Mul}\,(X) \simeq \coprod_{n \in \mathbb{N}} X^n/\mathfrak{S}_n$$

The failure of $\mathrm{Mul}\,$ to preserve pullbacks comes from the fact that quotients by group actions are "badly-behaved" in the category $\mathrm{Set}\,$, and this can be fixed by working homotopically (in that case going from sets to groupoids). However the failure of $\mathrm{Mul}\,$ to preserve finite products comes from the disjoint union: for instance, in the case of the nullary product, we have $\mathrm{Mul}\,(1) \simeq \mathbb{N} \not\simeq 1$ (and this would be the same without the quotients by symmetric groups). There is no reasonable way to fix this, as the very goal of the multiset functor in our setting is to model an exponential of linear logic: something that turns additives into multiplicatives cannot be expected to preserve multiplicatives.

Even though $\mathrm{Mul}\,:\mathrm{Set}\, \to \mathrm{Set}\,$ does not preserve finite products, it maps finite coproducts to finite products.

Fact 3.6.37. *The functor* $\mathrm{Mul}\,:\mathrm{Set}\, \to \mathrm{Set}\,$ *is strongly symmetric monoidal from the cocartesian structure to the cartesian structure on* $\mathrm{Set}\,$.

Proof. This is because $\mathrm{Mul}\,(X)$ is the free commutative monoid on X, so $\mathrm{Mul}\,$ is the composite monad of the adjunction between the category of sets and the category of commutative monoids. The left adjoint — the *free commutative monoid functor* — preserves finite coproducts by virtue of being a left adjoint. The category of commutative monoids has finite biproducts, i.e. its finite products are also coproducts and vice versa. Finally, the forgetful functor preserves finite products by virtue of being a right adjoint. In summary:

$$\mathrm{Set}\, \underset{\times \mapsto \times}{\overset{\sqcup \mapsto \sqcup}{\underset{\xrightarrow{\;\perp\;}}{\rightleftarrows}}} \mathrm{CMon}\,^{(\sqcup = \times)}$$

$\square$

Proposition 3.6.38. *The coproduct functor* $-\sqcup- : \mathrm{Set}\, \times \mathrm{Set}\, \to \mathrm{Set}\,$ *is regular.*

Proof. Consider diagrams

$$
\begin{array}{ccc}
& & Y \\
& & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\;f\;} & Z
\end{array}
\qquad\qquad
\begin{array}{ccc}
& & Y' \\
& & \downarrow{\scriptstyle g'} \\
X' & \xrightarrow{\;f'\;} & Z'
\end{array}
$$

in Set. The canonical map

$$\theta : X \times_Z Y \sqcup X' \times_{Z'} Y' \to (X \sqcup X') \times_{(Z \sqcup Z')} (Y \sqcup Y')$$
$$(x \in X, y \in Y) \mapsto (x \in X \sqcup X', y \in Y \sqcup Y')$$
$$(x' \in X', y' \in Y') \mapsto (x' \in X \sqcup X', y' \in Y \sqcup Y')$$

is clearly injective. Let $x \in X \sqcup X'$ and $y \in Y \sqcup Y'$ such that $(f \sqcup f')(x) = (g \sqcup g')(y)$. If $x \in X$, $(f \sqcup f')(x) = f(x) \in Z$, so that y must belong to Y, and reciprocally if $y \in Y$ then $x \in X$. Similarly, $x \in X'$ if and only if $y \in Y'$. In both cases, we have (x, y) is in the image of θ, so θ is bijective and $- \sqcup -$ preserves pullbacks.

Since it preserves pullbacks, it preserves kernel pairs. Moreover, colimits commute with colimits, the functor also preserves coequalizers, and hence regular epimorphisms (this can also be seen directly from the fact that the regular epis in Set are the surjections). □

In particular, we get another symmetric monoidal structure on Rel by lifting $- \sqcup -$. By fact 3.5.4, this symmetric monoidal structure is actually the cartesian monoidal structure on Rel.

Remark 3.6.39. The fact that the product and coproduct functors

$$- \times - : \mathsf{Set} \to \mathsf{Set}$$
$$- \sqcup - : \mathsf{Set} \to \mathsf{Set}$$

preserve pullbacks can be seen as a consequence of them being polynomial functors (see the discussion following theorem 5.8.26).

Combining proposition 3.6.38, fact 3.5.4, corollary 3.6.32, proposition 3.6.33 and fact 3.6.37, and the preservation of (symmetric) pseudomonoids by the 2-functor Rel $(-)$, we get the following.

Corollary 3.6.40. *The lifted multiset functor* Mul : Rel $\to$ Rel *is strongly symmetric monoidal from the cartesian symmetric monoidal structure on* Rel *to the tensor product symmetric monoidal structure.*

This gives us axiom (4) of the definition of Seely category (definition 3.4.3). We now turn our attention to item 3, i.e. the comonad structure on the lifted multiset functor Mul : Rel $\to$ Rel.

Proposition 3.6.41. *The monad laws*

$$\mu : \mathsf{Mul} \circ \mathsf{Mul} \Rightarrow \mathsf{Mul} \qquad \eta : \mathrm{id}_{\mathsf{Set}} \to \mathsf{Mul}$$

of the multiset monad on Set *are relational natural transformations.*

Proof. We work from the fibred point of view on multisets.

Relationality of η. Let X be a set. The map $\eta_X : X \to \mathsf{Mul}(X)$ maps $x \in X$ to $(1, \bar{x})$ where x denotes the map sending $1 \in \{1\}$ to $x \in X$. Given a map $f : X \to Y$, the naturality square

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
{\scriptstyle x \mapsto (1,\bar{x})} \downarrow & & \downarrow {\scriptstyle y \mapsto (1,\bar{y})} \\
\mathsf{Mul}(X) & \xrightarrow[(m,g) \mapsto (m, f \circ g)]{} & \mathsf{Mul}(Y)
\end{array}
$$

yields a canonical map

$$\theta : X \to \mathsf{Mul}(X) \times_{\mathsf{Mul}(Y)} Y$$
$$x \mapsto ((1, \bar{x}), \overline{f(x)})$$

By definition of the pullback in Set, and element $((m, g), y) \in \mathrm{Mul}(X) \times_{\mathrm{Mul}(Y)} Y$ must satisfy $(m, f \circ g) = (1, \bar{y})$, so $m = 1$, $g = \bar{x}$ for some $x \in X$ with $f(x) = y$. In particular, the triple $((m, g), y)$ has to be of the form $((1, \bar{x}), f(x))$, so θ is bijective and η_A is actually cartesian (and a fortiori relational).

Relationality of μ.   As before, given a map $f : X \to Y$, the naturality square

$$(m, (n_i, g_i)) \longmapsto (m, (n_i, f \circ g_i))$$

$$
\begin{array}{ccc}
(m, (n_i, g_i)) & \mathrm{Mul}(\mathrm{Mul}(X)) \longrightarrow \mathrm{Mul}(\mathrm{Mul}(Y)) & (m, (n_i, g_i)) \\
\updownarrow & \downarrow \qquad\qquad\qquad \downarrow & \updownarrow \\
(\sum_i n_i, [g_i]_i) & \mathrm{Mul}(X) \xrightarrow{\hspace{2cm}} \mathrm{Mul}(Y) & (\sum_i n_i, [g_i]_i)
\end{array}
$$

$$(m, g) \longmapsto (m, f \circ g)$$

yields a canonical map

$$\theta : \mathrm{Mul}(\mathrm{Mul}(X)) \to \mathrm{Mul}(X) \times_{\mathrm{Mul}(Y)} \mathrm{Mul}(\mathrm{Mul}(Y))$$

$$(m, (n_i, g_i)) \mapsto \left(\left(\sum_i n_i, [g_i]_i \right), \, (m, (n_i, f \circ g_i)) \right)$$

where the equality in $\mathrm{Mul}(Y)$ results from the equality $[f \circ g_i]_i = f \circ [g_i]_i$.

An element of $\mathrm{Mul}(X) \times_{\mathrm{Mul}(Y)} \mathrm{Mul}(\mathrm{Mul}(Y))$ is of the form

$$\left((m, g), (m', (n_i, h_i)) \right)$$

with $m = \sum_{i=1}^{m'} n_i$ and $f \circ g = [h_i]_i$. Choosing an isomorphism

$$\{1, \ldots, m\} \simeq \{1, \ldots, n_1\} \sqcup \cdots \sqcup \{1, \ldots, n_{m'}\}, \tag{3.4}$$

by universal property of the coproduct, the maps h_i are entirely determined by the choice of g, so there exist unique maps g_i such that $h_i = f \circ g_i$, and we have

$$\left((m, g), (m', (n_i, h_i)) \right) = \theta((m, (n_i, g_i)))$$

as desired. It might look like we showed that θ is bijective, but we had to make a choice of isomorphism (3.4), so this only shows surjectivity, hence μ is a relational natural transformation. $\qquad\square$

Remark 3.6.42. We can actually show that μ is not a cartesian natural transformation. Indeed, consider the unique map $f : \mathrm{Bool} \to 1$, and the multisets

$$m := [[0; 1]; [0]]$$
$$m' := [[0; 0]; [1]]$$

We have

$$\theta(m) = ([0; 0; 1], [[0; 0]; [0]]) = \theta(m')$$

even though $m \neq m'$, so θ is not injective and not an isomorphism. Going through the previous proof, we can see that the multisets m and m' are lifts of $([0;0;1],[[0;0];[0]])$ constructed from two different choices of isomorphisms

$$\{a,b,c\} \simeq \{a,b\} \sqcup \{c\}.$$

The issue stems once again from the "badly-behaved" quotients in Set, just like in remark 3.6.35. This issue does not occur for η since it lands in the $X^1/\mathfrak{S}_1$ term of the expression

$$\mathrm{Mul}\,(X) \simeq \coprod_{n \in \mathbb{N}} X^n/\mathfrak{S}_n,$$

and the action of $\mathfrak{S}_1$ on X^1 is always a *free action*. From the point of view of homotopy theory, group actions are always "free": if G acts on X, one can always "homotopically thicken" X so that the action becomes free, and in that sense homotopy quotients are better behaved than set-theoretical ones (see facts 6.6.28 and 6.6.29 and remark 6.11.22 for more precise ∞-categorical discussions of this phenomenon).

Corollary 3.6.43. *The lifted multiset functor* $\mathrm{Mul}\, : \mathrm{Rel}\, \to \mathrm{Rel}\,$ *inherits a monad structure. By self-duality of* $\mathrm{Rel}\,$ *(fact 3.6.12), this also gives a comonad structure on* $\mathrm{Mul}\, : \mathrm{Rel}\, \to \mathrm{Rel}\,$.

This gives us axiom (3) of the definition of Seely category (definition 3.4.3).

Axioms (1) and (2) don't follow from the same kind of justifications. The monoidal closure can be easily checked by showing that $X \times Y$ has the universal property of an internal hom object from X to Y in Rel. Axiom (2) is more interesting since the cartesian product in Rel comes from the coproduct in Set. What makes this work is that the canonical map $P(X) \times P(Y) \to P(X \sqcup Y)$ in Set is an isomorphism. Even though the construction $P : \mathrm{Set}\, \to \mathrm{Set}\,$ doesn't exist in an arbitrary regular category $\mathcal{C}$, one can always consider the full subcategory $\mathrm{Sub}\,(x) \subseteq \mathcal{C}_{/x}$ spanned by the monomorphisms. This category is equivalent to a partial order, and we can ask for a criterion for the canonical functor

$$\mathrm{Sub}\,(X) \times \mathrm{Sub}\,(Y) \to \mathrm{Sub}\,(X \sqcup Y)$$

to be an equivalence. A sufficient condition that is has already been well-developed in the literature is for $\mathcal{C}$ to be an *extensive category* [CLW93], i.e. that the canonical functor

$$\mathcal{C}_{/x} \times \mathcal{C}_{/y} \to \mathcal{C}_{/x \sqcup y}$$

is an equivalence for all $x, y \in \mathcal{C}$. Another sufficient condition is for $\mathcal{C}$ to have a subobject classifier Ω, in which case we have

$$\begin{aligned}
\mathrm{Sub}\,(x) \times \mathrm{Sub}\,(y) &\simeq \mathrm{Hom}_\mathcal{C}(x, \Omega) \times \mathrm{Hom}_\mathcal{C}(y, \Omega) \\
&\simeq \mathrm{Hom}_\mathcal{C}(x \sqcup y, \Omega) \\
&\simeq \mathrm{Sub}\,(x \sqcup y).
\end{aligned}$$

Both of these conditions happen for instance when $\mathcal{C}$ is an elementary topos, as is the case of Set.

This suggests that categories of internal relations in any topos could be equipped with a Seely structure. However I haven't yet found an abstract argument to show axiom (5) other than by direct computation.

3.7 From relations to spans

Given sets X and Y, a relation $R \subseteq X \times Y$ witnesses "whether or not" $x \in X$ and $y \in Y$ are related in some way: it is a qualitative notion. This is especially self-evident in the indexed point of view where a relation is seen as a map $r : X \times Y \to \mathrm{Bool}$. There the information contained in the boolean $r(x, y)$ is just "whether or not" x and y are related, it does not say anything about "how" they are related.

In the quest for more quantitative semantics of linear logic, the authors of [Lai+13] have generalized the relational model of linear logic to categories of *weighted relations*, replacing the role played by Bool in the index point of view by an arbitrary continuous semiring $\mathcal{R}$.

By trying to generalize the fibred point of view instead, we are led to replace relations $R \subseteq X \times Y$, i.e. monomorphisms $R \hookrightarrow X \times Y$, by arbitrary maps $R \to X \times Y$. Such a map is called a span from X to Y, and spans can be composed via pullback in Set

$$
\begin{array}{ccccc}
& & R \times_Y R' & & \\
& {\scriptstyle \pi_1}\swarrow & {\scriptstyle \vee} & \searrow{\scriptstyle \pi_2} & \\
& R & & R' & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \quad {\scriptstyle s'}\swarrow & & \searrow{\scriptstyle t'} \\
X & & Y & & Z
\end{array}
$$
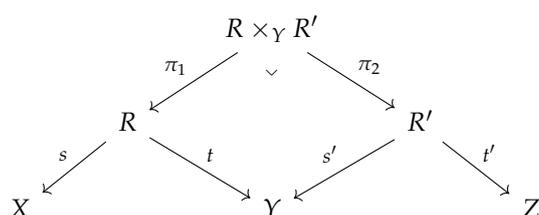
However this composition is not strictly associative, as pullbacks are associative only up to (natural) isomorphism. When defining categories of internal relations, we went around this issue by considering relations up to isomorphism. This was a rather innocuous choice, as the category $\mathrm{Sub}\,(X)$ is equivalent to a poset, so quotienting by isomorphisms was akin to quotienting a preorder to obtain a poset. In other words, the $(2, 2)$-category of sets and relations is equivalent to a $(1, 2)$-category, and the $(2, 1)$-category of sets and relations is equivalent to a category. But in the case of spans, there are in general many isomorphisms between spans from X to Y. Explicitly, the category of spans from X to Y is $\mathrm{Set}_{/X \times Y}$, which even in the simplest case $X = Y = \{*\}$ a singleton set, is equivalent to the category Set itself. Quotienting out by isomorphism of spans anyway would thus result in a loss of information. More precisely, a span up to isomorphism from X to Y is the same as a map from $X \times Y$ to Card, the (large) set of (small) cardinal numbers, so this would fall back to a category of weighted relations in the sense of [Lai+13] (with the added subtlety of the continuous semiring Card being large).

Definition 3.7.1. Let $\mathcal{C}$ be a category with pullbacks. The bicategory $\mathrm{SPAN}\,(\mathcal{C})$ of *spans in $\mathcal{C}$* has:

1. as objects the objects of $\mathcal{C}$,

2. as morphisms $x \twoheadrightarrow y$ the spans (r, s, t)

$$
\begin{array}{ccc}
& r & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\
x & & y
\end{array}
$$
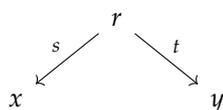
where s and t stand for "source" and "target",

3. as 2-morphisms $f : (r, s, t) \to (r', s', t')$ the maps $f : r \to r'$ making both following triangles

commute

$$
\begin{array}{c}
r \\
s \swarrow \quad \downarrow f \quad \searrow t \\
x \qquad \qquad y \\
s' \searrow \quad \downarrow \quad \swarrow t' \\
r'
\end{array}
$$
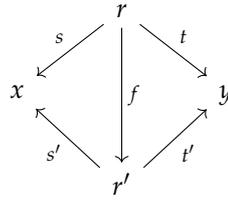
4. identities $\mathrm{id}_x : x \twoheadrightarrow x$ are given by

$$
\begin{array}{c}
x \\
\mathrm{id}_x \swarrow \qquad \searrow \mathrm{id}_x \\
x \qquad \qquad x
\end{array}
$$
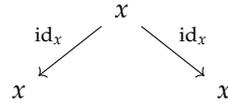
5. composition of $(r,s,t) : x \twoheadrightarrow y$ and $(r',s',t') : y \twoheadrightarrow z$ is given by

$$
\begin{array}{c}
r \times_y r' \\
\pi_1 \swarrow \quad \vee \quad \searrow \pi_2 \\
r \qquad\qquad\qquad r' \\
s \swarrow \quad \searrow t \quad s' \swarrow \quad \searrow t' \\
x \qquad\quad y \qquad\quad z
\end{array}
$$
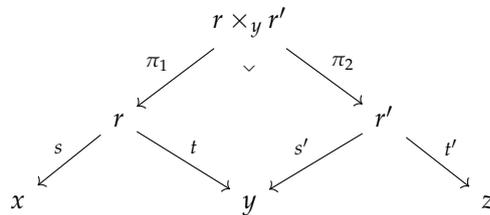
Restricting to only invertible 2-morphisms yields a $(2,1)$-category Span $(\mathcal{C})$.

We don't give the full definition of vertical and horizontal composition as we will treat those in more detail in section 5.6, our goal here will be to motivate the study of spans in homotopy type theory.

A first observation is that we only need $\mathcal{C}$ to have pullbacks for the definition of SPAN $(\mathcal{C})$ to make sense. This is in contrast with Rel $(\mathcal{C})$, where we also needed a well-behaved factorization system (although we could probably weaken the assumptions on $\mathcal{C}$ to only ask for weaker pullbacks). However, to lift a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor SPAN $(F) :$ SPAN $(\mathcal{C}) \to$ SPAN $(\mathcal{D})$, we *need F* to preserve pullbacks, in contrast to the weak preservation condition in the definition of relational functors (proposition 3.6.19). Indeed, to give an idea of the necessity, suppose that $\mathcal{C}$ admits a terminal object in addition to finite pullbacks, and let $f : x \to z, g : y \to z$ be morphisms in $\mathcal{C}$. Then preserving the pullback $x \times_z y$ is equivalent to preserving the following composition of spans:

$$
\begin{array}{c}
x \times_z y \\
\swarrow \quad \vee \quad \searrow \\
x \qquad\qquad y \\
\swarrow \quad \searrow f \quad g \swarrow \quad \searrow \\
1 \qquad\quad z \qquad\quad 1
\end{array}
$$
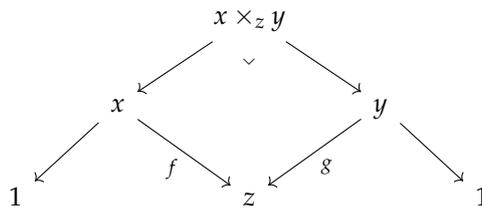
Similarly, to lift natural transformations to the world of spans, we need them to be cartesian.

Remark 3.7.2. The fact that cartesian monads lift to (co)monads on bicategories of spans has been known since at least [Bur71] where they were used to define multicategories. This perspective has since been expanded upon, notably in Leinster's book [Lei04].

Specifying to the case of $\mathcal{C} = \mathrm{Set}$, as explained in remarks 3.6.34 and 3.6.42, the multiset monad on Set does not preserve pullbacks, and its monad laws are not cartesian, so it cannot be lifted to a comonad on SPAN (Set).

One way to solve this problem is to generalize multisets in the same way we generalized relations: sticking with the fibred point of view on relations motivated us to consider spans as Set-valued relations rather than Card-valued relations. In the same way, rather than considering multisets as maps $X \to \mathbb{N}$ with finite support, we could consider maps from X to the category FinSet of finite sets. From the fibred point of view on multisets, this would mean considering multisets on X as pairs $(E \in \mathrm{FinSet}, f : E \to X)$ rather than pairs $(m \in \mathbb{N}, f : \{1, \ldots, m\} \to X)$. Then an isomorphism between multisets would be given by a bijection $\sigma : E \to E'$ such that the following triangle commutes:

$$
\begin{array}{ccc}
E & \xrightarrow[\sim]{\sigma} & E' \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle g} & \\
& X &
\end{array}
$$

By doing this, the "set" Mul (X) would no longer be a set: it would be a groupoid (this is precisely the HMul construction hinted at in remarks 3.6.35 and 3.6.36). In other words, we would need to consider the tricategory of spans in the bicategory of groupoids to make sense of this approach. But then the combinatorics of higher cells would quickly become too complicated to keep track of if we wanted to follow the same arguments as in section 3.6.

For instance, a span-version of theorem 3.6.27 would look like this:

Conjecture 3.7.3. *The construction $\mathcal{C} \mapsto \mathrm{SPAN}(\mathcal{C})$ defines a trifunctor from the bicategory of categories with pullbacks, pullback-preserving functors and cartesian natural transformations to the tricategory of bicategories.*

We only write this statement as a conjecture since we haven't checked all the conditions of being a trifunctor (notably the coherence for 3-cells). As we consider spans in bicategories, this would become even more hellish, as the analogous statement would be about a tetrafunctor between tetracategories. Interestingly, the monoidal tricategory of spans in a (strict) bicategory has been described by A.E.Hoffnung in [Hof13] in more than 130 pages. This gives an idea of how strenuous it would be to undertake a direct computational approach to proving that the SPAN construction defines a tetrafunctor.

From here on out there are various possible paths:

1. find a way to keep the discussion at a bicategorical level,

2. discard higher cells to survey the low-dimensional combinatorics of spans of groupoids,

3. find another point of view on categories of spans to make the higher combinatorics more approachable.

The first path has been explored for instance by Melliès using techniques from the theory of model categories [Mel19], and by Clairambault and Forest by restricting the spans considered to constrain higher morphisms [CF23]. The second approach is the content of chapter 5, in which we work in homotopy type theory to be able to easily access the low-dimensional combinatorics not only of groupoids, but of ∞-groupoids. The third approach can itself be divided in two:

- in section 3.8 we will see how an extensional point of view on relations and spans may lend itself better to such higher-categorical generalizations, and we'll see how to effectively do these generalizations in section 7.2,

- one could use a characterization of categories of spans through a universal property to get some functoriality for free (see open problem 7.2.32).

Open problem 3.7.4. Show that the $\mathcal{C} \mapsto \mathrm{SPAN}\,(\mathcal{C})$ construction is functorial in a higher-categorical sense.

3.8 The extensional point of view on Rel

As noticed in remark 3.5.3, the formula giving the composition of relations in the indexed point of view is very reminiscent of the formula for matrix composition. This observation is but one aspect of a general analogy at play: in the same way matrices are an intensional presentation of an extensional concept (linear maps between vector spaces), relations are also an intensional representation for another extensional, namely linear maps between suplattices. In this section we explore in more detail the extensional point of view on relations. The general analogy is summed up in table 3.4.

Remark 3.8.1. We got the idea to use the terms intensional and extensional to refer to this dichotomy from M. Fiore, Z. Galal and H. Paquet's article on *Stabilized profunctors and stable species of structures* [FGP23]. An *extensional definition* is a definition that "lists" every object that falls under it, while an *intensional definition* specifies its instances by giving necessary and sufficient conditions. For instance, the set

$$X = \{x^2 + 4x + 1 \mid x \in \mathbb{Z}\}$$

is described extensionally by listing every number it contains, but it can also be defined intensionally[1] as

$$\{x \in \mathbb{N} \mid x + 3 \text{ is a perfect square}\}.$$

In the setting of linear algebra, linear maps are an extensional concept, since a linear map f is characterized by all the possible pairs $(x, f(x))$ that constitute its graph, while a matrix is an intensional representation of how to compute such a linear map.

Notation 3.8.2. Write $P(X)$ for the powerset (i.e. the set of subsets) of a set X.

Fact 3.8.3. *The partial order $(P(X), \subseteq)$ is a complete lattice : it admits arbitrary joins and arbitrary meets, respectively given by unions and intersection of sets.*

Let $R \subseteq X \times Y$ be a relation. It induces a map between powersets

$$(\!|R|\!) : P(X) \to P(Y)$$
$$(U \subseteq X) \mapsto \{y \in Y \mid \exists x \in U, x\,R\,y\}$$

Fact 3.8.4. *The map $(\!|R|\!) : P(X) \to P(Y)$ preserves arbitrary joins.*

Definition 3.8.5 ([JT84])**.** A monotone map $f : (E, \leq) \to (F, \leq)$ between complete lattices is called *linear* if it preserves arbitrary joins. The category of complete lattices and linear maps is written SupLat and its objects are called *suplattices*.

Remark 3.8.6. The "sup" in "suplattice" is a shorthand for "supremum", which is another word for join.

---

[1]For readers interested in formal set theory, extensional definitions of sets correspond to uses of the axiom schema of replacement, while intensional definitions correspond to uses of the axiom schema of specification.

Remark 3.8.7. By definition, a suplattice is just a complete lattice, but when using the former term we think of the morphisms as preserving only arbitrary joins, while the latter we think of the arbitrary meets as a part of the structure too. In general a linear map between suplattices may not preserve any meets.

Remark 3.8.8. The arbitrary joins of elements in a suplattice are analogous to the sums of vectors in a vector space. The suplattice $P(X)$ is the *free suplattice* generated by the set X, in a way analogous to how $\text{Vect}_k(X)$ is the free k-vector space generated by the set X. The linear map $(\!|R|\!) : P(X) \to P(Y)$ induced by a relation $R : X \nrightarrow Y$ is analogous to the linear map between vector spaces induced by a matrix between their bases.

Remark 3.8.9. Suplattices were first defined and studied by Joyal and Tierney [JT84] in the context of the theory of frames and locales. Frames are suplattices where finite meets distribute over arbitrary joins, hence in their setting they play a role analogous to the role of rings or algebras with respect to the vector space role played by suplattices.

Fact 3.8.10. *The mapping*

$$X \mapsto P(X) \tag{3.5}$$

$$R \mapsto (\!|R|\!) \tag{3.6}$$

defines a functor $P : \text{Rel} \to \text{SupLat}$. This functor is moreover fully faithful.

Spelled out explicitly, the fully faithfulness here means that every linear map

$$f : P(X) \to P(Y)$$

between powerset suplattices is uniquely of the form $(\!|R|\!)$ for some relation $R : X \nrightarrow Y$. In particular, Rel is equivalent to the full subcategory of SupLat on the powerset suplattices.

Remark 3.8.11. In linear algebra, every vector space is free on some basis, so the free vector space functor $\text{Vect}_k(-)$ is actually an equivalence between the category k-Mat of sets and k-valued matrices and the category k-Vect of k-vector spaces and linear maps. Comparatively, not every suplattice is free, so the functor from relations to suplattices is only fully faithful and not essentially surjective. This is similar to what happens when one considers categories of modules over a ring rather than vector spaces over a field: there, not every module is in general free.

This embedding of Rel into SupLat preserves a lot of the structures and properties of Rel, as exemplified by the few following facts.

Fact 3.8.12. *The category* SupLat *admits finite biproducts (i.e. finite products and coproducts, and they coincide), and they are computed as the products of the underlying sets. Moreover, the embedding $P : \text{Rel} \hookrightarrow \text{SupLat}$ preserves finite products and coproducts.*

Remark 3.8.13. Note how the products and coproducts in Rel are computed as the disjoint union of underlying sets, while they are computed as the products of the underlying sets in SupLat. That the embedding Rel $\hookrightarrow$ SupLat preserves products and coproducts then means that on underlying sets, we have canonical isomorphisms

$$P(X \sqcup Y) \simeq P(X) \times P(Y),$$

correspond to the fact that a subset of $X \sqcup Y$ is uniquely determined by a pair of a subset of X and a subset of Y.

In linear algebra, this corresponds to the fact that the cartesian product (or direct sum) of vector spaces has for basis the disjoint union the bases of its factors.

Fact 3.8.14. *The category* SupLat *is symmetric monoidal closed, with a tensor product $\otimes$ characterized by the following universal property: for every $X, Y, Z \in$ SupLat, there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{SupLat}}(X \otimes Y, Z) \simeq \mathrm{Bilin}(X \times Y, Z)$$

natural in X, Y and Z, where $\mathrm{Bilin}(X \times Y, Z)$ denotes the set of maps $f : X \times Y \to Z$ that are bilinear, i.e. such that for every $x \in X$, $f(x, -) : Y \to Z$ is linear and for every $y \in Y$, $f(-, y) : X \to Z$ is linear.
  The internal hom is the poset $\mathrm{Hom}_{\mathrm{SupLat}}(X, Y)$ under pointwise ordering.

Fact 3.8.15. *The embedding $P :$ Rel $\hookrightarrow$ SupLat preserves the tensor product and the internal hom (it's a symmetric monoidal closed functor).*

Remark 3.8.16. The isomorphism

$$P(X \times_{\mathrm{Set}} Y) \simeq P(X) \otimes P(Y)$$

should be reminiscent of the fact from linear algebra that the tensor product $E \otimes F$ of two vector spaces has as basis the cartesian product of the bases of E and F. The isomorphism

$$P(X \times_{\mathrm{Set}} Y) \simeq P(X) \multimap P(Y)$$

on the other hand, corresponds to the encoding of matrices in $M_{m,n}(k)$ as vectors of length $m \times n$.

Fact 3.8.17. *The category* SupLat *is $*$-autonomous, with dualizing object the two-elements suplattice* Bool $:= \{0 < 1\}$. *The P embedding preserves the dualizing object:*

$$\{0 < 1\} \simeq \{\varnothing \subsetneq \{\star\}\} = P(\{\star\})$$

Fact 3.8.18. *The category* SupLat *admits cofree commutative comonoids, and the P embedding preserves cofree commutative comonoids.*

Combining all of the above, we get:

Fact 3.8.19. SupLat *is a Lafont $*$-autonomous with finite products and coproducts, hence a full model of classical linear logic, in which the relational model embeds fully faithfully.*

In particular, it would have been possible to work with suplattices from the very beginning, derive all the Lafont and $*$-autonomous structure there, and then notice that it restricts to free suplattices to get the usual relational model as a "submodel" of SupLat. This may seem inconsequential now, but when we will get to the ∞-categorical counterpart to the relational model in section 7.2, we will work exclusively through this extensional perspective, because even simply defining higher categories of relations intensionally is way more complicated than to do so extensionally (remark 7.2.33). This extensional perspective on relations can also bring to light other interesting ways to model the exponential comonad, as we're about to see in the next section.

3.9 Free suplattices and the model of domains

Recall the existence of another Seely structure on Rel where the exponential ! is modeled by the powerset comonad P (proposition 3.5.18). In the extensional perspective, this corresponds to the following:

---

[1]Here we moreover need the assumption that for every $y \in Y$, there are only finitely many $x \in X$ such that $M_{x,y} \neq 0$. This is because linear algebra is concerned with finite sums, while the theory of partial orders makes sense of arbitrary joins.

Relations	Vector spaces
Bool	Base field k
Set X	Basis X
Relation $r : X \times Y \to$ Bool	Matrix[1] $M : X \times Y \to k$
Identity relation $\mathrm{id}_X : (x,y) \mapsto \delta_x^y$	Identity matrix $M_{x,y} = \delta^y$
Composition formula $(r;r')(x,z) = \bigvee_{y \in Y} r(x,y) \wedge r'(y,z)$	Matrix multiplication $(M \cdot N)_{x,z} = \sum_{y \in Y}(M_{x,y} \times N_{y,z})$
Predicate $p : X \to$ Bool	Vector $v : X \to k$ (with finite support)
Free suplattice $P(X)$	Free vector space $\mathrm{Vect}_k(X)$
Join-preserving map $P(X) \to P(Y)$	Linear map $\mathrm{Vect}_k(X) \to \mathrm{Vect}_k(Y)$

Table 3.4: Analogy between relations and matrices.

Fact 3.9.1. *There is a linear/non-linear adjunction*

$$\mathrm{Set} \underset{|-|}{\overset{P}{\underset{\perp}{\rightleftarrows}}} \mathrm{SupLat}$$

Where we write $|X|$ for the underlying set of a suplattice X.

By definition, the functor $P : \mathrm{Set} \to \mathrm{SupLat}$ lands in the full subcategory of powerset suplattices, so it restricts to the functor $P : \mathrm{Set} \to \mathrm{Rel}$ of proposition 3.5.18. The induced comonad $P(|-|)$ is simply the powerset comonad when thought of as a comonad on Rel, and when thought of as a comonad on SupLat it is the "free suplattice on the underlying set" comonad.

Remark 3.9.2. The above adjunction also induces the usual powerset monad on Set. Interestingly, the category Rel can also be defined as the Kleisli category for this monad, as any relation $R : X \nrightarrow Y$ can be equivalently described as a function $f : X \to P(Y)$, i.e. a Kleisli morphism for P. On the other hand, the structure of a suplattice on a set X is exactly the structure of an algebra for the powerset monad: given such a structure $\mu : P(X) \to X$, we can define an ordering on X by $x \leq y$ if $\mu(\{x,y\}) = y$, the algebra axioms guarantee that μ computes the join of subsets for this ordering. In other words, SupLat can be seen as the category of algebras — or Eilenberg-Moore category — of the powerset monad P on Set, and the above adjunction is monadic in that sense.

Rather than considering the "free suplattice on the underlying set" comonad, we can also consider the "free suplattice on the underlying **poset**". Given a poset E, the set of downward-closed subsets of E ordered by inclusion is a suplattice, and it is moreover the free suplattice on E. When E is a discrete poset on a set, all its subsets are downward-closed. Because of this we extend the notation $P(X)$ to posets:

Definition 3.9.3. The *powerset of a poset E* is the poset $P(E)$ of its downward-closed subsets.

Remark 3.9.4. In the indexed point of view on subsets (section 2.6.1), a downward-closed subset $U \subseteq E$ corresponds to a monotonous map $\chi_U : E^{\mathrm{op}} \to \{0 < 1\}$. From the point of view of category theory, such a map is a Bool-enriched presheaf on E, and the fact that the poset of these maps is the free suplattice on E is a decategorification of the statement that the Yoneda embedding $\yen : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is the free cocompletion of $\mathcal{C}$.

Fact 3.9.5. *There is a linear/non-linear adjunction*

$$\text{Poset} \xrightleftharpoons[\perp]{P} \text{SupLat}$$

where the functor SupLat $\hookrightarrow$ Poset *is the inclusion of the subcategory of suplattices in the category of posets.*

The induced exponential comonad P on SupLat computes the free suplattice on a suplattice. Contrarily to $P(|-|)$, this comonad on SupLat does not restrict to Rel, as the free suplattice on a given suplattice E is not in general free on a set (this is the case only when E is the discrete ordering on its underlying set, which since E is a suplattice forces E to be a singleton). However, P can clearly be restricted to the full subcategory of SupLat on suplattices of the form $P(E)$ for E a poset, and this category admits a nice intensional description similar to that of Rel.

Definition 3.9.6. *Let E, F be posets. A posetal relation $E \nrightarrow F$ is a relation $R \subseteq E \times F$ such that $\forall x, x' \in E, \forall y, y' \in F$,*

$$x' \geq x \, R \, y \geq y' \implies x' \, R \, y'.$$

In the indexed point of view on subsets, a posetal relation $E \nrightarrow F$ corresponds to a map $E \times F^{\text{op}} \to$ Bool.

Definition 3.9.7. The category Porel of posets and *posetal relations* has posets as objects and posetal relations as morphisms.

Fact 3.9.8. *Every posetal relation $R : E \nrightarrow F$ induces a morphism of suplattices*

$$(\!|R|\!) : P(E) \to P(F)$$
$$(U \subseteq E) \mapsto \{y \in Y \mid \exists x \in U, x \, R \, y\},$$

(where U is any downward-closed subset of E, and the fact that $(\!|R|\!)(U)$ is downward-closed is guaranteed by R being a posetal relation).
  Moreover, the induced functor $P :$ Porel $\to$ SupLat is fully faithful.

Hence we have a chain of full and faithful embeddings Rel $\hookrightarrow$ Porel $\hookrightarrow$ SupLat.

Fact 3.9.9. *The closed symmetric monoidal, cartesian and cocartesian structures on* SupLat *restrict to* Porel *via isomorphisms*

$$P(E) \otimes P(F) \simeq P(E \times F) \tag{3.7}$$
$$P(E) \multimap P(F) \simeq P(E^{\text{op}} \times F) \tag{3.8}$$
$$\text{Bool} \simeq P(\{\star\}) \tag{3.9}$$
$$P(E) \times_{\text{SupLat}} P(E) \simeq P(E \sqcup F) \tag{3.10}$$
$$P(E) \sqcup_{\text{SupLat}} P(E) \simeq P(E \sqcup F) \tag{3.11}$$

Notice the only difference with fact 3.5.5 being that we need to be careful to define the internal hom in Porel as $E^{\text{op}} \times F$ rather than $X \times Y$ like in Rel.

Fact 3.9.10. *The exponential comonad induces by the LNL adjunction of fact 3.9.5 restricts to the "free suplattice on a poset" comonad on* Porel.

The previous comonad $P(|-|)$ also restricts to Porel, as well as the Lafont exponential.

Fact 3.9.11. *The category* Porel *admits a Lafont structure, where the cofree commutative comonoid on a poset E is given by the free posetal commutative monoid E, i.e. the set of multisets on E ordered by*

$$[a_1, \ldots, a_n] \leq [b_1, \ldots, b_m]$$

whenever $n = m$ and there exists $\sigma \in \mathfrak{S}_n$ such that $b_i = a_{\sigma(i)}$. This poset is written $\mathrm{PMul}\,(E)$.

The comonads $P(|-|)$ and P on SupLat both come from free/forgetful adjunctions between SupLat and categories of objects with way less structure: the forgetful functor SupLat $\hookrightarrow$ Poset forgets all the joins that existed in a suplattice, while the forgetful functor SupLat $\to$ Set forgets even forgets the very ordering between elements. There is another adjunction that exhibits a less extreme forgetting of structure: that between suplattices and *Scott domains*.

Remark 3.9.12. The following few results are not proven here, as we will explain in detail how their ∞-categorical counterparts work in section 7.2.2, and their 0-categorical (i.e. posetal) version here follow from similar arguments in a way simpler setting.

Definition 3.9.13. In a poset E, a *directed subset* is a subset $U \subseteq E$ such that

- $U \neq \varnothing$,

- $\forall x, y \in U, \exists z \in U, x \leq z$ and $y \leq z$.

More succinctly, it is a subset $U \subseteq E$ such that all of its finite subsets admit an upper bound in U.

Definition 3.9.14. A *Scott domain* is a poset in which every directed subset admits a join. We write Scott for the category of Scott domains and monotonous maps that preserve the joins of directed subsets.

Fact 3.9.15. *There is a linear/non-linear adjunction*

$$\mathrm{Scott} \; \underset{\perp}{\overset{L}{\rightleftarrows}} \; \mathrm{SupLat}$$

The left adjoint L : Scott $\to$ SupLat freely adjoins joins to a Scott domain E while preserving the existing joins of directed subsets.

To understand how the induced exponential comonad L : SupLat $\to$ SupLat acts on suplattices of the form $P(E)$, we use a few abstract results on the interaction between joins of directed subsets and joins of finite subsets. First, we have a chain of left adjoints to the forgetful functors SupLat $\hookrightarrow$ Scott $\hookrightarrow$ Poset :

$$\mathrm{Poset} \; \underset{\perp}{\overset{S}{\rightleftarrows}} \; \mathrm{Scott} \; \underset{\perp}{\overset{L}{\rightleftarrows}} \; \mathrm{SupLat}$$

Since adjoints compose, the left adjoint of the forgetful functor SupLat $\hookrightarrow$ Poset is the composite of the left adjoints L and S to the forgetful functors SupLat $\hookrightarrow$ Scott and Scott $\hookrightarrow$ Poset . In other words, we have $P(E) = L(S(E))$ for any poset E.

Now let E be a poset, and write $E^\vee$ for the free cocompletion[1] of E under *finite joins*.

Fact 3.9.16. *Given a poset E, we have $P(E) \simeq S(E^\vee)$. In other words, taking the free Scott domain of the finite-join cocompletion of a poset E yields its cocompletion under arbitrary joins $P(E)$.*

---

[1] We use the term *cocompletion* here and not completion to emphasize that joins are a posetal version of colimits, and to reinforce the analogy with the 1-categorical and ∞-categorical settings (respectively section 3.11 and section 7.2).

In particular, we have

$$L(P(E)) = L(S(E^\vee)) = P(E^\vee)$$

Corollary 3.9.17. *The exponential comonad* S : SupLat $\to$ SupLat *restricts to an exponential comonad on* Porel *which acts on objects as* $E \mapsto E^\vee$. *This comonad is called the* Scott *exponential.*

Remark 3.9.18. The poset $E^\vee$ can be explicitly described in at least two ways:

1. as the set of finite subsets of E ordered by

$$\{a_1, \ldots, a_n\} \le \{b_1, \ldots, b_m\}$$

   whenever there exists some $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $\forall i, a_i \le b_{f(i)}$.

2. as the posetal quotient[1] of the preorder on Mul (E) given by

$$[a_1, \ldots, a_n] \le [b_1, \ldots, b_m]$$

   whenever there exists some $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $\forall i, a_i \le b_{f(i)}$.

The second definition may seem needlessly complicated when compared to the first, but it is easier to compare to the free exponential PMul on Porel: the only difference between the two is that in PMul (E), multisets can only be compared when they have the same size. For instance, in $E^\vee$ we have $[x] \le [x, y]$, while in PMul (E) these elements are incomparable.

As a consequence, in $E^\vee$ we have $[x, x] = [x]$, while in PMul (E) we have $[x, x] \ne [x]$. In other words, duplication in $E^\vee$ is *idempotent*, and in PMul (E) it is *non-idempotent*.

Remark 3.9.19. Following the previous remark, it has been shown by Thomas Ehrhard that the Scott exponential is formally the collapsing of the PMul exponential in [Ehr12]. Interestingly, in his article he mentions that working in the intensional perspective of posets and posetal relations is easier to describe the constructions of linear logic in the model. One of the points of this thesis is to demonstrate how on the contrary, the extensional perspective can be way easier to work with in more abstract settings: as we will see in section 7.2, the ∞-categorical counterpart to SupLat is easily definable, while giving a direct definition to Porel's counterpart is a whole challenge of its own (see remark 7.2.33).

Summing everything up, there is a chain of symmetric monoidal left adjoints

$$\text{Set} \xrightarrow[\bot]{} \text{Poset} \xrightarrow[\bot]{S} \text{Scott} \xrightarrow[\bot]{L} \text{SupLat}$$

where the monoidal structures on Set, Poset and Scott are cartesian, yielding three different linear/non-linear adjunctions on the category SupLat, and hence three models for the exponential ! of intuitionistic linear logic. These three comonads compute three kinds of cocompletions:

1. $P(|E|)$ the free cocompletion of the underlying set of E under arbitrary joins (i.e. the free suplattice on the underlying set of E),

2. $P(E)$ the free cocompletion of the poset E under arbitrary joins (i.e. the free suplattice on the poset E),

---

[1] Here, the "posetal quotient" of a preorder means quotienting by $x \sim y$ whenever both $x \le y$ and $y \le x$: it's the universal way to turn a preorder into a poset.

3. $E^\vee$ the free cocompletion of the poset E under finite joins.

The Lafont exponential PMul on the other hand, is not a free cocompletion in the categorical sense. It is the only one of these four comonads on Porel which models the duplication aspect of the exponential ! in a non-idempotent way (see remark 3.9.18)

Remark 3.9.20. The $E \mapsto E^{\mathrm{op}}$ self-duality of the category Porel means we could have considered posetal relations with an opposite variance, i.e. $E^{\mathrm{op}} \times F \to \mathrm{Bool}$ rather than $E \times F^{\mathrm{op}} \to \mathrm{Bool}$. In that setting, the Scott exponential would compute the free completion under *finite meets* rather than *finite joins*. This convention is for instance followed in Olimpieri's thesis [Oli20, Section 2.7], but we stick to the choice we made here as we find it more natural, as well as more intuitive for the generalizations of sections 3.11 and 7.2.

3.10 Models from linear algebra

Throughout section 3.8, we entertained an analogy between the relational model of linear logic and linear algebra. One may justly wonder if an analogous model of ILL may be built from a category of vector spaces, and indeed that is the case, in more than one way.

Remark 3.10.1. This section is mostly independent from the rest of this thesis, apart from the models in *higher linear algebra* we develop in section 7.3. I nonetheless decided to include it as I found it quite hard to navigate the literature on vector space-based models of linear logic, and I hope the following exposition and the few references provided there may be of help to future students and researchers interested in getting some core intuitions and ideas.

Rather than working specifically in vector spaces over a field k, we work in the generality of modules over a commutative ring R, of which we recall the definition here:

Definition 3.10.2. Let R be a commutative ring. A *R-module* (or simply module when R is clear from context) M is an abelian group $(M, +, 0)$ together with an external multiplication law

$$- \cdot - : R \times M$$

such that for all $r, r' \in R$ and $m, m' \in M$,

$$
\begin{aligned}
1_R \cdot m &= m & (r \times r') \cdot m &= r \cdot (r' \cdot m) \\
0_R \cdot m &= 0_M & (r + r') \cdot m &= (r \cdot m) + (r' \cdot m) \\
r \cdot 0_M &= 0_M & r \cdot (m + m') &= (r \cdot m) + (r \cdot m')
\end{aligned}
$$

The morphisms of R-modules are the morphisms of abelian groups $f : M \to M'$ such that $f(r \cdot m) = r \cdot f(m)$ for every $r \in R, m \in M$.

We write Mod_R for the category of R-modules and their morphisms.

When $R = k$ is a field, an R-module is by definition the same thing as a k-vector space: the categories Mod_k and Vect_k are equivalent. When $R = \mathbb{Z}$ the ring of integers, every abelian group admits a unique $\mathbb{Z}$-module structure, and every abelian group morphism is a morphism of $\mathbb{Z}$-modules, so $\mathrm{Mod}_{\mathbb{Z}}$ is equivalent to the category of abelian groups.

We now turn to the interpretation of the logical connectives of linear logic in Mod_R:

Fact 3.10.3. *The category* Mod_R *is symmetric monoidal closed, with a tensor product characterized by the following universal property:*

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M \otimes_R M', N) \simeq \mathrm{Bilin}_R(M \times M', N)$$

where $\mathrm{Bilin}(M \times M', N)$ *denotes the set of* bilinear maps, *i.e. maps* $f : M \times M' \to N$ *such that for every* $m \in M$ *and* $m' \in M'$ *the maps* $f(m, -) : M' \to N$ *and* $f(-, m') : M \to N$ *are R-module morphisms.*

The unit for the tensor product is the base ring R seen as an R-module (the external multiplication being given by the internal multiplication of the ring).

The internal hom is given by the set $\mathrm{Hom}_{\mathrm{Mod}_R}(M, N)$ *where the R-module structure is computed pointwise.*

Here we see the strong similarity with the closed symmetric monoidal structure on SupLat (fact 3.8.14). The theme of "the tensor product classifying bilinear maps" will play a very important role again in section 7.2.1.

Fact 3.10.4. *The category* Mod $_R$ *admits finite biproducts.*

Remark 3.10.5. The category Mod $_R$ actually admits (small) products and coproducts, but only the finite products coincide with the finite coproducts. On the other hand, the category Rel admits small biproducts, even though Rel can be thought of as a sort of "category of free modules and matrices over the semiring Bool". This discrepancy follows from the fact that the semiring Bool is *infinitary*: one can compute arbitrary "sums" of elements in Bool, while only finite sums are allowed in R-modules.

Just like in Rel, Porel or SupLat, there are multiple ways to represent the exponential ! in Mod $_R$. First of all, we have the following counterpart to proposition 3.5.18:

Proposition 3.10.6. *The forgetful functor* $| - | : \mathrm{Mod}_R \to \mathrm{Set}$ *admits a left adjoint* $R\langle - \rangle : \mathrm{Set} \to \mathrm{Mod}_R$ *that fits in a linear/non-linear adjunction:*

$$(\mathrm{Set}, \times) \xrightarrow[\substack{\longleftarrow \\ |-|}]{R\langle - \rangle} (\mathrm{Mod}_R, \otimes_R)$$

The left adjoint $R\langle - \rangle$ computes the *free R-module* on a set X: this can be expressed explicitly as a direct sum

$$R\langle X \rangle := \bigoplus_{x \in X} R,$$

using the symbol $\oplus$ instead of $\sqcup$ to denote coproducts in categories of R-modules, as this is the more common notation in linear algebra.

Remark 3.10.7. That this left adjoint is symmetric monoidal from $(\mathrm{Set}, \times)$ to $(\mathrm{Mod}_R, \otimes_R)$ means in particular that the free module on $X \times Y$ is the tensor product of the free modules on X and Y:

$$R\langle X \times Y \rangle \simeq R\langle X \rangle \otimes_R R\langle Y \rangle$$

For vector spaces, this is the well-known fact that a basis for the tensor product $E \otimes_k F$ is given by a cartesian product of bases for E and F (already mentioned in remark 3.8.16).

The model of Proposition 3.10.6 does not seem to be widely known in the linear logic community, or at least not widely mentioned. For instance the authors of the standard reference [VZ14] restrict themselves to the category of finite-dimensional vector spaces over a finite field to better understand concretely the computational behaviour of the model, but do not mention that the model works even for infinite-dimensional vector spaces and for modules over commutative rings.

Others focus instead on the more complicated free exponential ([Mur17],[HS03, Section 2.4]), but they are also restrict themselves to working with vector spaces and not more general modules, even though the following holds:

Proposition 3.10.8. *The symmetric monoidal category* $(\mathrm{Mod}_R, \otimes_R)$ *admits cofree commutative comonoids, and thus forms a Lafont category.*

In linear algebra, (co)monoids with respect to $\otimes_R$ are more usually called (co)algebras, so the above result would be spelled "categories of modules admit cofree commutative coalgebras".

For categories of vector spaces over a field k, this result dates back to Moss E. Sweedler's 1969 monograph on Hopf Algebras [Swe69, Section 6.4], where he gives an explicit construction. More recently, Daniel Murfet reviewed Sweedler's construction and gave a new one in the case where the base field k is algebraically closed of characteristic 0 [Mur15]. For more general categories of R-modules, the existence of cofree commutative coalgebras has been shown in 1973 by Michael Barr using an adjoint functor theorem [Bar74]. To the author's knowledge, no explicit construction of the cofree commutative coalgebra is known for modules over a general commutative ring.

Because the existence of cofree commutative coalgebras in categories of modules has been known for more than 50 years, the only really new result in proposition 3.10.8 is the framing of this existence as defining a Lafont structure on general categories of modules.

Linear algebra and $*$-autonomy. An important difference between the category Rel of sets and relations and categories Mod_R of modules over a ring is that the latter is in general *not* $*$-autonomous.

Proposition 3.10.9. *Let R be a commutative ring. The category Mod_R is $*$-autonomous if and only if R is the zero ring $R = \{0\}$, in which case Mod_R is a contractible category (i.e. contains a unique object up to unique isomorphism).*

We give a proof here for lack of a better reference.

Proof. We rely on the theory of locally presentable categories, of which we will study the ∞-categorical counterpart in section 6.8. The category Mod_R is locally presentable, being the category of models for an algebraic theory. Suppose Mod_R is $*$-autonomous: in particular, $\mathrm{Mod}_R \simeq \mathrm{Mod}_R^{\mathrm{op}}$, and so $\mathrm{Mod}_R^{\mathrm{op}}$ must also be locally presentable. By [Adá94, Thm 1.64], a category $\mathcal{K}$ for which both $\mathcal{K}$ and $\mathcal{K}^{\mathrm{op}}$ are locally presentable must be equivalent to a poset. So Mod_R must be equivalent to a poset.

We show this implies R to be the zero ring in the following way: considering R as an R-module in the evident way, the two projection maps

$$\pi_1 : R \times R \to R$$
$$\pi_2 : R \times R \to R$$

must be equal in $\mathrm{Hom}_{\mathrm{Mod}_R}(R \times R, R)$ since Mod_R is equivalent to a poset. This means that given $(x, y) \in R \times R$, we have $x = y$, so the underlying set of R is a singleton, and R is the zero ring.

Finally, it is a straightforward exercise to check that a $\{0\}$-module structure on an abelian group A forces A to be the trivial group, so Mod_R only contains one object (up to unique isomorphism). $\square$

The above proof is rather abstract. To get more intuition into this failure of $*$-autonomy, we can look at the analogy between Mod_R and Rel or SupLat. In Rel and in SupLat, the dualizing object is given by the unit for the tensor product. In Mod_R, that unit is the R-module R, and in general we call the module $(M \multimap R)$ the *dual* of the module M. Given an R-module M, the canonical R-module morphism

$$M \to (M \multimap R) \multimap R$$

is given by

$$m \mapsto (\phi \mapsto (\phi(m)))$$

There are multiple ways in which this map can fail to be an isomorphism.

Consider for instance $R := \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \multimap \mathbb{Z}$ is trivial, since the only abelian group morphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ is the trivial one, and so its dual is also trivial:

$$(\mathbb{Z}/2\mathbb{Z} \multimap \mathbb{Z}) \multimap \mathbb{Z} = 0 \neq \mathbb{Z}/2\mathbb{Z}$$

Remark 3.10.10. More generally, taking the double dual of a finitely generated abelian group M yields its torsion-free subgroup, i.e. the subgroup of elements without torsion (an element *with* torsion is an element $x \in M$ such that $n \cdot x = 0$ for some $n \in \mathbb{Z} \setminus \{0\}$).

This suggests looking at cases where there is no torsion at all: vector spaces over a field k. Here we can say in general that the map $M \to (M \multimap k) \multimap k$ is injective, but it is a well-known result from linear algebra that it is not surjective in general:

Fact 3.10.11. *Given a k-vector space V, the canonical map $V \to (V \multimap k) \multimap k$ is an isomorphism if and only if k is finite dimensional.*

Yet again we can understand the issue by comparing with Rel: every k-vector space admits a basis, so the category of k-vector spaces and linear maps is equivalent to a category with sets as objects and "matrices" as morphisms. More precisely, if X is a basis for E and Y a basis for F, a linear map $f : E \to F$ is determined by the expression of the elements $(f(x))_{x \in X}$ as *finite* weighted sums of elements of Y:

$$f(x) = \sum_{y \in Y} a_{x,y} \cdot y$$

with $a_{x,y} \in k$. This can be represented as an "infinite matrix" $F : X \times Y \to k$, with the crucial property that for every $x \in X$, there is only a finite number of $y \in Y$ such that $F(x, y) \neq 0$: in other words, the rows of the matrix must have *finite support*. In particular, swapping the rows and columns no longer necessarily yields a matrix: not every linear map $E \to F$ corresponds to a linear map $F \to E$.

By contrast, in Rel, there is no such finiteness conditions in the matrix representation of a relation as a map $r : X \times Y \to \text{Bool}$, since Bool admits arbitrary joins of elements, and not only finite joins. Because of that, a relation $r : X \times Y \to \text{Bool}$ is readily seen to be equivalent to a relation $r : Y \times X \to \text{Bool}$.

There are three main ways to go around this issue:

1. Equipping vector spaces with additional structures to restrict linear maps sufficiently as to get a $*$-autonomous category: this is for instance the approach of Thomas Ehrhard's *Köthe sequence spaces* approach [Ehr02].

2. Embrace the need for infinite sums and consider (semi)rings in which such sums exist, this was notably the motivation for the *weighted relational models* of Laird, Manzonetto, McCusker and Pagani [Lai+13].

3. Building on fact 3.10.11, trying to find exponential comonads directly on the category of finite dimensional k-vector spaces. Here the Lafont comonad does not exist in general, but the $k\langle | - | \rangle$ comonad exists when k is a finite field: this is studied in [VZ14]. The authors of [HS03] also mention the possibility of defining an exponential comonad based on the *exterior algebra* when k is of characteristic 2.

3.11 Models in profunctors and species

As explained in the introduction, in the past few years, various authors have investigating bicategorical models of linear logic. We already mentioned span-based models in section 3.7, generalizing the relational model. In this section, we mention another bicategorical model, that we will generalize in section 7.2: the model of profunctors and species, developed in a series of articles since 2007 by various authors [Fio+07; Fio14; Fio+17; FGP23; FGH24].

We introduce this model here by analogy with the relational model.

Definition 3.11.1. A *profunctor* $F : \mathcal{C} \nrightarrow \mathcal{D}$ between small categories $\mathcal{C}$ and $\mathcal{D}$ is a functor

$$F : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}\,.$$

The morphisms of profunctors are the natural transformations.

Seeing posets as Bool -enriched categories, profunctors are to categories what posetal relations are to posets. Just like posetal relations can be presented extensionally as suplattices morphisms $P(E) \to P(F)$, profunctors admit an extensional presentation.

Notation 3.11.2. We write $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}\,)$ the category of presheaves of sets on a small category $\mathcal{C}$.

Remark 3.11.3. By curryfication, a profunctor $F : \mathcal{C} \nrightarrow \mathcal{D}$ can be equivalently described as a functor

$$F : \mathcal{C} \to \mathcal{P}(\mathcal{D}),$$

in quite the same way that a posetal relation $R : E \times F^{\mathrm{op}} \to \mathrm{Bool}$ is the same thing as a monotonous map $R : E \to P(F)$.

Fact 3.11.4. *The characterization of presheaf categories as universal cocompletions of small categories yields for every small categories $\mathcal{C}, \mathcal{D}$ an equivalence of categories*

$$\mathrm{Fun}\,(\mathcal{C}, \mathcal{P}(\mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{cc}}\,(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D}))$$
$$F \mapsto \mathrm{Lan}_{\sharp}\,(F)$$
$$G \circ \sharp \leftarrow\!\shortmid G$$

where $\mathrm{Fun}_{\mathrm{cc}}\,(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D}))$ *denotes the category of cocontinuous functors from $\mathcal{P}(\mathcal{C})$ to $\mathcal{P}(\mathcal{D})$, and* $\mathrm{Lan}_{\sharp}\,(F)$ *denotes the left Kan extension along the Yoneda embedding*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{P}(\mathcal{D}) \\
{\scriptstyle \sharp}\downarrow & \nearrow & \\
\mathcal{P}(\mathcal{C}) & {\scriptstyle \mathrm{Lan}_{\sharp}\,(F)} &
\end{array}
$$

Fact 3.11.5. *There is a bicategory, denoted* Prof *, whose objects are small categories, whose morphisms are profunctors and whose 2-morphisms are the natural transformations. The composition of profunctors $F : \mathcal{C} \nrightarrow \mathcal{D}$, $G : \mathcal{D} \nrightarrow \mathcal{E}$ is given by the coend formula*

$$(G \circ F)(c, e) := \int^{d \in \mathcal{D}} F(c, d) \times G(d, e),$$

and the identities are the hom-profunctors

$$\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$
$$(c, c') \mapsto \mathrm{Hom}_{\mathcal{C}}\,(c', c)$$

This leads us to the following bicategorical counterpart to fact 3.9.8:

Fact 3.11.6. *Write* Cat_{cc} *for the bicategory of cocomplete categories, cocontinuous functors and natural transformations. The presheaf constructions* $\mathcal{C} \mapsto \mathcal{P}(\mathcal{C})$, *together with the equivalence of fact 3.11.4 yield a fully-faithful 2-functor*

$$\mathrm{Prof} \hookrightarrow \mathrm{Cat}_{cc}.$$

In other words, Prof *is equivalent to the full sub-bicategory of* Cat_{cc} *on the categories of the form* $\mathcal{P}(\mathcal{C})$ *for some small category* $\mathcal{C}$.

Intuitionistic linear logic is modeled in Prof in the following way:

Fact 3.11.7. Prof *admits a closed symmetric monoidal structure given by*

- $\mathcal{C} \otimes_{\mathrm{Prof}} \mathcal{D} := \mathcal{C} \times_{\mathrm{Cat}} \mathcal{D}$

- $\mathbf{1}_{\mathrm{Prof}}$ *is the terminal category* 1

- $\mathcal{C} \multimap \mathcal{D} := \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$

Prof *is moreover* $*$*-autonomous with dualizing object* $\mathbf{1}$, *since*

$$(\mathcal{C} \multimap \mathbf{1}) = (\mathcal{C}^{\mathrm{op}} \times \mathbf{1}) \simeq \mathcal{C}^{\mathrm{op}}$$

and $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Fact 3.11.8. *The category* Prof *admits (small) biproducts, given by coproducts in* Cat *(i.e. disjoint union of categories).*

Just like for posetal relations and suplattices, these structures also exist at the level of Cat_{cc}. The following result is not especially well-known in the 1-categorical community, and I don't know an explicit reference for it, but it can be seen as a corollary of a more general ∞-categorical result we will review in section 7.2 (see remark 7.2.6).

Fact 3.11.9. Cat_{cc} *admits a closed symmetric monoidal structure, where the tensor product* $\mathcal{C} \otimes \mathcal{D}$ *classifies functors* $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ *that are cocontinuous independently in both variables. The monoidal closure is given by*

$$(\mathcal{C} \multimap \mathcal{D}) := \mathrm{Fun}_{cc}(\mathcal{C}, \mathcal{D}).$$

Remark 3.11.10. Even though the $*$-autonomy of Porel extends to SupLat, that is not the case for Prof and Cat_{cc}: the object $\mathcal{P}(1) = \mathrm{Set}$ is not dualizing in Cat_{cc}.

Indeed, consider for instance the category Ab of abelian groups. It is a cocomplete category, and the category $\mathrm{Fun}_{cc}(\mathrm{Ab}, \mathrm{Set})$ is trivial: let $F : \mathrm{Ab} \to \mathrm{Set}$, be a cocontinuous functor. F must map the initial object in Ab (the trivial group 1) to the initial object in Set (the empty set $\varnothing$). But since 1 is also terminal in Ab, for every abelian group A, the unique morphism $A \to 1$ must be mapped by F to a map $F(A) \to \varnothing$, so that $F(A)$ must also be the empty set. Since this holds for every $A \in \mathrm{Ab}$, the functor F must be the constant functor at the empty set. In the end, we get the following chain of equivalences:

$$\mathrm{Fun}_{cc}(\mathrm{Fun}_{cc}(\mathrm{Ab}, \mathrm{Set}), \mathrm{Set}) \simeq \mathrm{Fun}_{cc}(1, \mathrm{Set}) \simeq 1,$$

and $1 \neq \mathrm{Ab}$.

Remark 3.11.11. The above example of $\text{Fun}_{cc}(\text{Ab}, \text{Set}) \simeq 1$ is reminiscent of the example

$$(\mathbb{Z}/2\mathbb{Z} \multimap \mathbb{Z}) \simeq 1$$

from the discussion preceding remark 3.10.10. Thinking of the bicategory Cat_{cc} as a categorification of the category Ab, the object $\text{Ab} \in \text{Cat}_{cc}$ looks like a "torsion object" in that its dual with respect to the unit Set is trivial.

Remark 3.11.12. One reason why SupLat is $*$-autonomous and Cat_{cc} isn't is that in SupLat, the dual of an suplattice L is its opposite suplattice:

$$(L \multimap \text{Bool}) \simeq L^{\text{op}}.$$

This makes sense because every suplattice necessarily admits arbitrary meets (the meet of a subset being the join of its complement).

On the other hand, the dual of cocomplete category in Cat_{cc} has no reason to be its opposite: the opposite of a cocomplete category is in general not cocomplete. For presheaf categories for instance, we have

$$\text{Fun}_{cc}(\mathcal{P}(\mathcal{C}), \text{Set}) \simeq \text{Fun}(\mathcal{C}, \text{Set}) \simeq \mathcal{P}(\mathcal{C}^{\text{op}}),$$

and $\mathcal{P}(\mathcal{C}^{\text{op}}) \not\simeq \mathcal{P}(\mathcal{C})^{\text{op}}$. This last inequation would be true if we had an equivalence $\text{Set} \simeq \text{Set}^{\text{op}}$, but that is not the case. In SupLat on the other hand, we do have an equivalence $\text{Bool} \simeq \text{Bool}^{\text{op}}$, given by negation.

The additive connectives also extend to Cat_{cc}:

Fact 3.11.13. *The category* Cat_{cc} *admits (small) biproducts, and they are computed as cartesian products of the underlying categories.*

This is sound with the computation of biproducts in Prof as coproducts in Cat (fact 3.11.8), since we have

$$\begin{aligned}
\mathcal{P}(\mathcal{C} \sqcup \mathcal{D}) &\simeq \text{Fun}(\mathcal{C}^{\text{op}} \sqcup \mathcal{D}^{\text{op}}, \text{Set}) \\
&\simeq \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set}) \times \text{Fun}(\mathcal{D}^{\text{op}}, \text{Set}) \\
&\simeq \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{D})
\end{aligned}$$

Now remains the question of how to interpret the exponential ! on Prof and/or on Cat_{cc}.

3.11.1 The Lafont exponential, generalized species and normal functors.

The first description we did of the multiset comonad on Rel was as a lifting of the multiset monad on Set, which computes the free commutative monoid on a set (facts 3.5.14 and 3.5.15). In 2007, Fiore, Gambino, Hyland and Winskel showed that the same thing could be achieved for Prof:

Fact 3.11.14 ([Fio+07]). *The* free symmetric monoidal category *monad* $\text{Sym} : \text{Cat} \to \text{Cat}$ *lifts to a monad on* Prof, *which by* $*$-*autonomy of* Prof *then yields a comonad on* Prof.

Note that the *free symmetric monoidal category* $\text{Sym}(\mathcal{C})$ on a category $\mathcal{C}$ is exactly the *free commutative monoid* on $\mathcal{C}$ in Cat in the suitable bicategorical sense.

Remark 3.11.15. The category $\text{Sym}(\mathcal{C})$ can be described explicitly as a category whose objects are lists of objects of $\mathcal{C}$, and the morphisms exist only between lists of the same size $(c_1, \ldots, c_n) \to (d_1, \ldots, d_n)$ and are of the form

$$(\sigma, (f_1, \ldots, f_n))$$

where

- $\sigma \in \mathfrak{S}_n$,

- $\forall 1 \leq i \leq n, f_i \in \mathrm{Hom}_{\mathcal{C}}(c_i, d_{\sigma}(i))$.

This is once again the same construction as one discussed before conjecture 3.7.3.

More recently, they have shown that this comonad actually computed the *free commutative comonoid* in Prof, so Sym : Prof $\to$ Prof is actually a bicategorical free exponential in the sense of Lafont [FGH24, Thm 8.1]. Again, this is to be compared with the similar statement for Rel (remark 3.5.17).

Remark 3.11.16. The proof of fact 3.11.14 in [Fio+07] relies on results about distributive laws between (co)monads on bicategories, developed in [Fio+17]. When we will get to the ∞-categorical counterpart of this result in section 7.2.3, we will instead be working directly at the extensional level, i.e. in Cat$_{\mathrm{cc}}$ rather than in Prof, and use a very general existence criteria for the existence of free commutative monoids to build the free exponential, and only after show that it computes as expected (section 7.2.3 and theorem 7.2.26). It would be interesting to reach the same result using the ∞-categorical version of distributive laws for monads introduced by Kositsyn in [Kos21], but we leave that for future work.

In their original 2007 article [Fio+07], Fiore et. al. were actually mainly interested not in the Sym comonad on Prof itself, but in its Kleisli category.

Definition 3.11.17. The non-linear morphisms

$$F \in \mathrm{Hom}_{\mathrm{Prof}_{\mathrm{Sym}}}(\mathcal{C}, \mathcal{D})$$

are called *generalized species of structure* from $\mathcal{C}$ to $\mathcal{D}$, or simply generalized species.

Unfolding the definition, a generalized species F from $\mathcal{C}$ to $\mathcal{D}$ can be described as a functor

$$F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{P}(\mathcal{D}).$$

Using the explicit description of Sym $(\mathcal{C})$ given in remark remark 3.11.15, the authors of [Fio+07] gave the following extensional description of generalized species:

Definition 3.11.18 ([Fio+07]). Let $F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ be a functor. It induces a functor $\widetilde{F} : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ given by the coend formula

$$\widetilde{F}(X)(d) := \int^{(c_1, \ldots, c_n) \in \mathrm{Sym}(\mathcal{C})} F((c_1, \ldots, c_n)) \times X(c_1) \times \cdots \times X(c_n) \tag{3.12}$$

The functors that arise in this way are called *analytic functors*.

In the simple case that $\mathcal{C} = \mathcal{D} = 1$ is the terminal category, the analytic functors Set $\to$ Set are precisely those that admit a description of the form

$$X \mapsto \bigsqcup_{n \in \mathbb{N}} (A_n \times X^n)/\mathfrak{S}_n, \tag{3.13}$$

where the $(A_n)_{n \in \mathbb{N}}$ are a sequence of sets equipped with actions of the symmetric groups $\mathfrak{S}_n$. This kind of expression is a categorification of the classical expression of analytic functions

$$f(x) = \sum_{n \in \mathbb{N}} \frac{a_n x^n}{n!},$$

and the more general formula (3.12) above is akin to a multi-variable generalization of (3.13).

Remark 3.11.19. Analytic functors from Set to Set were first defined and applied by André Joyal to the field of combinatorics, categorifying the already well-established use of analytic functions there [Joy81]. The generalized species and analytic functors of Fiore extended the notion to general functors between presheaf categories.

Analytic functors are very similar to the normal functors studied by Girard in [Gir88], with two crucial differences:

1. Girard only considered functors between categories of the form Set^X, where X is a set instead of a more general category.

2. The definition of normal functors cannot involve quotients, for instance the normal functors from Set to Set must be of the form

$$\bigsqcup_{n \in \mathbb{N}} A_n \times X^n.$$

Girard noticed that his bicategory of normal functors was not cartesian closed [Gir88, Rmk 2.19], but nonetheless managed to define a model of the λ-calculus. The bicategory of categories and generalized species, on the other hand, is cartesian closed (this is proven directly in [Fio+07], but can also be seen as a consequence of a bicategorical version of remark 3.4.5): in that sense, generalized species and analytic functors provide exactly the right generalization of normal functors to get a cartesian closed bicategory, and more generally an actual model of linear logic.

Remark 3.11.20. To make this more formal, we need to restrict to generalized species between groupoids: in his 2014 paper [Fio14], Fiore has shown that when $\mathcal{C}$ and $\mathcal{D}$ are groupoids, we get an equivalence

$$\text{Fun}\,(\text{Sym}\,(\mathcal{C}), \mathcal{P}(\mathcal{D})) \simeq \text{AF}\,(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D})),$$

where $\text{AF}\,(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D}))$ denotes the subcategory of $\text{Fun}\,(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D}))$ on analytic functors and *quasi-cartesian natural transformations*.

Writing AF for the bicategory of groupoids and analytic functors between their presheaf categories, we get an equivalence

$$\text{GpdProf}_{\text{Sym}} \simeq \text{AF},$$

where GpdProf denotes the full sub-bicategory of Prof on the groupoids, and $\text{GpdProf}_{\text{Sym}}$ the Kleisli category for the Sym comonad restricted to GpdProf.

The cartesian closure of Prof_{Sym} restricts to $\text{GpdProf}_{\text{Sym}}$, so the above equivalence implies that the bicategory AF of analytic functors is cartesian closed.

In section 5.8, we will use a different generalization of normal functors to obtain models of linear logic and cartesian closed (higher) categories, see section 5.8.3 for an explicit comparison.

3.11.2   The Scott exponential(s)

Just as the Mul comonad on Rel has a counterpart Sym on Prof (and GpdProf), the Scott exponential on Porel also has a counterpart on Prof. Such a counterpart was first studied by Cattani and Winskel in [CW05b], and further developed by Galal in her thesis [Gal21] where she gave another natural way to categorify the Scott exponential.

The idea behind both these generalizations is to find a suitable categorifications for the *finite joins* and the *directed joins* used to define Scott domains. Cattani and Winskel's approach was to replace *finite joins* with *finite colimits* and *directed joins* with *filtered colimits*, while Galal instead

replaced *finite joins* with *finite coproducts* and *directed joins* with *sifted colimits* (see [ARV10] for a short survey on sifted colimits).

There are multiple equivalent ways to define both filtered and sifted diagrams, we chose here the definitions that most closely match the definition of directed subsets (definition 3.9.13).

Definition 3.11.21. A category $\mathcal{C}$ is *filtered* if for every diagram $D : J \to \mathcal{C}$ with J a *finite category* admits a cocone, i.e. an object $c \in \mathcal{C}$ together with a natural transformation

$$D \Rightarrow \Delta c,$$

where $\Delta c : D \to \mathcal{C}$ denotes the constant diagram at the object c.

Definition 3.11.22. A category $\mathcal{C}$ is *sifted* if

- it is inhabited,

- for every pair of objects $c, c' \in \mathcal{C}$, the category of cocones for the diagram

$$\begin{aligned} \mathrm{Bool} &\to \mathcal{C} \\ 0 &\mapsto c \\ 1 &\mapsto c' \end{aligned}$$

  is connected.

Here a category being connected means in particular that it must be inhabited, so the definition can be rephrased as:

- for every finite *discrete* diagram $D : J \to \mathcal{C}$, the category of cocones on D is connected.

The following characterization of filtered and sifted categories is usually taken as a definition:

Fact 3.11.23. *A category $\mathcal{C}$ is filtered (respectively sifted) if $\mathcal{C}$-indexed diagrams in* Set *commute with finite limits (respectively, finite products).*

We can now state results analogous to fact 3.9.15 and corollary 3.9.17.

Notation 3.11.24. Given a category $\mathcal{C}$, we write $\mathcal{C}^{\sqcup}$ for the free cocompletion of $\mathcal{C}$ under finite coproducts, and $\mathcal{C}^{\mathrm{fc}}$ for its free cocompletion under finite sums.

Notation 3.11.25. We write $\mathrm{Cat}_{\mathrm{filtr}}$ (respectively $\mathrm{Cat}_{\mathrm{sift}}$) for the bicategory of categories which admit colimits of filtered (respectively, sifted) diagrams, and functors that preserve these colimits.

Fact 3.11.26. *There are linear/non-linear adjunctions of bicategories*

$$(\mathrm{Cat}_{\mathrm{filtr}}, \times) \; \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \; (\mathrm{Cat}_{\mathrm{cc}}, \otimes)$$

$$(\mathrm{Cat}_{\mathrm{sift}}, \times) \; \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \; (\mathrm{Cat}_{\mathrm{cc}}, \otimes)$$

The two induced comonads on $\mathrm{Cat}_{\mathrm{cc}}$ *restrict through the faithful embedding* $\mathrm{Prof} \hookrightarrow \mathrm{Cat}_{\mathrm{cc}}$, *and the induced action on the objects of* Prof *is respectively given by* $(-)^{\mathrm{fc}}$ *and* $(-)^{\sqcup}$.

Remark 3.11.27. In her thesis, Galal uses the notations $\mathcal{S}, \mathcal{F}$ and $\mathcal{C}$ instead of $\mathrm{Sym}, (-)^{\mathrm{fc}}$ and $(-)^{\sqcup}$ respectively.

Actually, we have a bigger chain of symmetric monoidal left adjoints

$$(\mathrm{Gpd}\,,\times)\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ (\mathrm{Cat}\,,\times)\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ (\mathrm{Cat}_{\mathrm{filtr}}\,,\times)\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ (\mathrm{Cat}_{\mathrm{sift}}\,,\times)\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ (\mathrm{Cat}_{\mathrm{cc}}\,,\otimes)$$

very similar to the chain

$$\mathrm{Set}\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ \mathrm{Poset}\ \underset{\longleftarrow}{\overset{S}{\underset{\perp}{\longrightarrow}}}\ \mathrm{Scott}\ \underset{\longleftarrow}{\overset{L}{\underset{\perp}{\longrightarrow}}}\ \mathrm{SupLat}$$

of section 3.8. We will say more about how to obtain this chain in the ∞-categorical setting in section 7.2.2. Since we have not mentioned size issues at all yet, it may come as a surprise to some reader that there is an actual adjunction

$$(\mathrm{Cat}\,,\times)\ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\ (\mathrm{Cat}_{\mathrm{cc}}\,,\otimes),$$

since the free cocompletion is usually constructed as the category of presheaves on a small or locally small category, and $\mathcal{P}(\mathcal{C})$ is not small (resp. locally small) when $\mathcal{C}$ is. However, it is always possible to construct the free cocompletion of a large category $\mathcal{C}$ without increasing its size, it will simply not be given by $\mathcal{P}(\mathcal{C})$. Once again, we will see this in more detail in the context of ∞-categories in fact 7.2.3.

Remark 3.11.28. Given a category $\mathcal{C}$, its free coproduct cocompletion $\mathcal{C}^{\sqcup}$ admits an explicit construction:

- the objects of $\mathcal{C}^{\sqcup}$ are lists of elements of $\mathcal{C}$,

- the morphisms $(c_1,\ldots,c_n) \to (d_1,\ldots,d_m)$ are given by tuples

$$(f,(g_1,\ldots,g_n))$$

  where $f : \{1,\ldots,n\} \to \{1,\ldots,m\}$ and $g_i \in \mathrm{Hom}_{\mathcal{C}}(c_i,d_{f(i)})$ for every $1 \leq i \leq n$.

This description is to be compared with the analogous description of the Scott exponential on posets (remark 3.9.18) and the explicit description of the Sym comonad on Prof (remark 3.11.15).

In a poset, joins are idempotent, so the Scott exponential creates idempotent duplications of elements, while the free exponential Mul doesn't. In the categorified setting however, neither Sym nor $(-)^{\sqcup}$ create idempotent duplications, since freely generated coproducts are not idempotent.

3.12   Differential linear Logic

A common aspect of all the models we have studied so far (Rel , Porel ,SupLat ,Mod $_R$,Prof ,Cat$_{\mathrm{cc}}$) is that they admit *biproducts*: the finite products coincide with finite coproducts. From the point of view of linear logic, this means that the additive connectives & and $\oplus$ are interpreted as one and the same:

$$[\![A \mathbin{\&} B]\!] \simeq [\![A \oplus B]\!].$$

In the literature, such models are often called *degenerate*.

Various methods exist for creating non-degenerate models from degenerate ones, for instance relying on *gluing* and *orthogonality constructions* [HS03], also studied in the bicategorical context in Galal's thesis [Gal21]. These methods can be used to describe models such as *coherence spaces*, *finiteness spaces*, *probabilistic coherence spaces* ([Lopa]). We will not go into more detail about these

non-degenerate models in this thesis, and will rather be interested in additional structures to be found in the *degenerate* ones: *differential structures*.

Differential linear logic (or DiLL for short) was introduced in the 2000's by Thomas Ehrhard and Laurent Reignier [ER03; ER06; Ehr18], following the observation that some models of linear logic supported well-behaved notions of differentiation. In models of linear logic, every linear map $f : X \multimap Y$ can be seen as a non-linear map $!X \multimap Y$ by pre-composing with the unit for the comonad $!$, $\eta : !X \multimap X$. The idea of differentiation is that every non-linear map $f : !X \multimap Y$ should admit a *linear approximation* $\partial f : X \multimap Y$ (for instance think of the differential at 0 $d_0 f$ of a smooth map normed vector spaces).

We won't go into the details of the syntax of DiLL in this thesis, instead noting that categorical semantics for it have been given under the concept of *differential categories* [BCS06; Blu+20]. Differential categories are actually more general than models of DiLL, since they do not require their "linear categories" to have a monoidal closure $\multimap$.

Various equivalent axiomatizations of differential categories are provided in [Blu+20], but when restricted to the case of a linear category $(\mathcal{C}, \otimes, \multimap, !)$ with biproducts, the structure of differential category simplifies to the data of a natural *codereliction map*

$$\bar{\mathsf{d}}_X \in \mathrm{Hom}_{\mathcal{C}}(X, !X)$$

subject to the commutativity of some diagrams.

Example 3.12.1. The differential structure on Rel was first studied in [Ehr05] (a more modern survey can be found in [Ehr18]), the codereliction map being given by the opposite relation of the counit for the comonad Mul on Rel :

$$\bar{\mathsf{d}}_X = \{(x, [x]) \mid x \in X\} \in \mathrm{Rel}\,(X, \mathrm{Mul}\,(X)).$$

Example 3.12.2. A differential structure on the Lafont category of k-vector spaces has been described in [CM20] using Sweedler's explicit description of the cofree commutative coalgebra (see section 3.10).

The two above examples can be seen as corollaries of a more general result by J. S. Lemay:

Fact 3.12.3 ([Lem21])**.** *Every Lafont category admits a structure of differential category.*

From this and proposition 3.10.8, we can actually derive the following corollary, more general than example 3.12.2, and yet seemingly not mentioned in the linear logic literature:

Corollary 3.12.4. *The category* Mod_R *of modules on a commutative ring R admits a structure of differential category.*

Remark 3.12.5. It has been observed in [BLO16] that the *opposite* of the category of k-vector spaces, $\mathrm{Vect}_k^{\mathrm{op}}$, has a structure of (non monoidal closed) differential category, with the free exponential in $\mathrm{Vect}_k^{\mathrm{op}}$. By definition, the cofree commutative coalgebra in $\mathrm{Vect}_k^{\mathrm{op}}$ is the free commutative algebra in Vect_k, often written Sym. The algebras for this monad are precisely the k-algebras in the usual sense of linear algebra, so the coalgebras for the corresponding comonad in $\mathrm{Vect}_k^{\mathrm{op}}$ form the opposite of the category of k-algebras.

In algebraic geometry, this category is notoriously equivalent to the category of *affine k-schemes*, and this differential structure on $\mathrm{Vect}^{\mathrm{op}}$ seem to be related to the usual notions of differentiation in scheme theory.

Open problem 3.12.6. It would be interesting to study the relation between the differential structures on Vect_k and $\mathrm{Vect}_k^{\mathrm{op}}$, or more generally on Mod_R and $\mathrm{Mod}_R^{\mathrm{op}}$.

DiLL in bicategories. Semantics of differential linear logic have very recently been considered in a bicategorical context by Fiore, Gambino and Hyland in [FGH24]. Instead of just generalizing the usual notion of differential category one dimension up, they found an elegant axiomatization for the codereliction map in terms of 2-cells.

First, they define a bicategorical counterpart to the notion of linear categories. Then, starting from a *linear bicategory* $(\mathcal{C}, \otimes, \multimap, !)$ in which the dereliction maps $\mathsf{d} : !A \to A$ (i.e. the counit of the comonad $!$) have left adjoints $\bar{\mathsf{d}} : A \to !A$, they give sufficient conditions for these lefts adjoints to be codereliction maps in the sense of [Blu+20].

One big advantage of their definition is that it avoids the standard issue of higher coherences. To explain what this means, consider for instance Bierman's axiom in the definition of Seely categories: this axiom takes the form of a specific diagram being commutative. If one were to give a bicategorical version of this axiom, this commutativity should instead be the data of an invertible 2-morphism between compositions of 1-morphisms, which itself should be subject to additional coherence conditions, and it is quite unclear in general how to minimally specify such higher coherences.

In differential categories, various diagrams involving the codereliction maps are required to commute in order to properly model DiLL. Rather than try to find suitable higher coherences for these diagrams in a linear bicategory $\mathcal{C}$, the definition of [FGH24] uses the unit and the counit of the defining adjunction between the codereliction map $\bar{\mathsf{d}}_A$ and the dereliction map d_A

$$A \xrightarrow[\; \mathsf{d}_A \;]{\overset{\bar{\mathsf{d}}_A}{\underset{\perp}{\longleftrightarrow}}} !A$$

to define suitable coherence 2-morphisms that are then simply required to be invertible.

Let's look at an example to make this idea clearer. Let $(\mathcal{C}, \otimes, \multimap, !)$ be a linear bicategory, and write $\mathsf{d}_A : !A \to A$ for the dereliction map (i.e. the counit of the comonad $!$). Unfolding the definitions of adjoint 1-morphisms in a bicategory, we are asking for the existence for every $A \in \mathcal{C}$ of a 1-morphism $\bar{\mathsf{d}}_A : A \to !A$ and of 2-morphisms

such that the zig-zag identities hold:

In that case η is called the unit of the adjunction $\bar{\mathsf{d}}_A \dashv \mathsf{d}_A$, and ε is its counit, and the triple $(\mathsf{d}_A, \eta, \epsilon)$ is unique up to isomorphism.

One of the axioms of differential categories is that there is a commutative triangle

$$
\begin{array}{ccc}
A & \xrightarrow{\;\bar{\mathsf{d}}_A\;} & !A \\
 & {\scriptstyle \mathrm{id}_A}\searrow & \downarrow{\scriptstyle \mathsf{d}_A} \\
 & & A
\end{array}
$$

In the axiomatization of [FGH24], this is not replaced by asking for an arbitrary invertible 2-morphism subject to further coherences. Instead it is replaced by asking for the unit $\eta : \mathrm{id}_A \Rightarrow \mathsf{d}_A \circ \bar{\mathsf{d}}_A$ to be an invertible 2-morphism: the *data* of an invertible 2-morphism is replaced by the *property* of a canonical 2-morphism being invertible. For the other axioms of differential categories, the authors of [FGH24] define other 2-morphisms using η, ε and other existing bicategorical structures, that are then required to be invertible in a similar way.

Using this approach, they managed to show that the linear bicategory

$$
(\mathrm{Prof}, \otimes, \multimap, \mathrm{Sym})
$$

is a bicategorical model of DiLL. In section 5.9, we will use the same axiomatization to show that our homotopy type theoretic model of linear logic based on spans of types is also a model of DiLL.

Chapter 4

Homotopy Type Theory

As explained in the discussion following conjecture 3.7.3, one possible approach to higher categorical models of linear logic is to discard higher coherences and look directly for homotopical structures that model linear logic at least at the level of their low-dimensional cells. The setting of homotopy type theory lends itself particularly well to this kind of consideration.

Homotopy type theory (HoTT) is an extension of Martin-Löf type theory (MLTT) defined and studied in the eponymous book [Uni13], that can serve as an alternative foundation of mathematics to ZFC set theory, and is particularly well-suited to the study of homotopy theory. Indeed, while the objects of study of set theory are sets, the objects of study of HoTT are the so-called *homotopy types*. This is a play on words as the term homotopy type historically refers to the equivalence class of a topological space under the relation of being "homotopy equivalent" (definition 2.2.13), but in this setting the word "type" also refers to the types of type theory. As explained in section 2.2, topological spaces up to homotopy equivalence correspond to ∞-groupoids, and as such, homotopy type theory is a mathematical foundation where the basic objects are ∞-groupoids instead of sets.

As we will see, every construction that can be done in HoTT is *homotopy-invariant*, in the sense that it respects the homotopical structure of ∞-groupoids: for instance every map between types will be functorial and equalities between maps will be natural. In that sense, homotopy type theory can be thought of as an *internal language for homotopy theory*, and this intuition has actually been formalized by showing that the formal system of homotopy type theory admits models in any ∞-topos [Shu19] and in particular in classical homotopy theory. Because of this, HoTT has also been described as the setting of *synthetic homotopy theory* [Buc19, Section 3.1].

Even though homotopy type theory is concerned with such complicated objects and concepts, it turns out to be a remarkably easy framework to work with. At a base level, it is just an extension of Martin Löf's theory of dependent types, which serves as the basis for many proof assistants (Rocq, Agda, Lean...), and at first it may be surprising that the basic constructions we will do carry any meaningful homotopical content, but that is precisely what makes working in HoTT so powerful.

In this chapter we give an overview of homotopy type theory, briefly motivating type theory in section 4.1, then giving various examples of how to define and use types in sections 4.2 and 4.3. This leads us naturally to consider identity/equality types in section 4.4, which we study further in section 4.5 through various examples, motivating the univalence axiom in section 4.6. The rest of the chapter is dedicated to reviewing concepts from homotopy theory in the setting of homotopy type theory and how to work with them: we present homotopy n-types in section 4.7, how to reason with equivalences in section 4.8, higher inductive types in section 4.10 (i.e. types

Logical formula or connective	Type or type-former
True proposition $\top$	Unit type 1
False proposition $\bot$	Empty type $\varnothing$
Conjunction $A \wedge B$	Product type $A \times B$
Disjunction $A \vee B$	Coproduct type $A \sqcup B$
Implication $A \implies B$	Function type $A \to B$
Predicate $P(x)$ on $x \in X$	Type family $P : X \to \mathcal{U}^1$
Existential quantifier $\exists x \in X, P(x)$	Dependent sum type $\sum_{x:X} P(x)$
Universal quantifier $\forall x \in X, P(x)$	Dependent function type $\prod_{x:X} P(x)$

Table 4.1: Correspondence between logical formulas and type-formers.

with constructors for equalities), truncations of types in section 4.11, and finally we apply all of this to the definition of homotopy groups in section 4.12.

4.1 Introduction

In ZFC set theory, everything is a set, and sets can be compared via two relations: equality $x = y$ and belonging $x \in y$. These two relations are then related by axioms postulated in first-order logic. It is *in that first-order logic* that the actual theorems and proofs of the theory are stated.

In type theory, the basic objects are called *types $A, B, \ldots$*. Just like sets, types can have elements, and we write $x : A$ to say "x is an element of type A". A big difference with set theory is that an element $x : A$ is not itself a type, and that an element cannot belong to different types.

While in set theory, the actual theory is done in first-order-logic, in type theory on the other hand, there is no distinction between the level of objects and the level of theorems and proofs. A type A can be thought of both as "thing with elements" or as a "proposition", in which case an element $x : A$ is thought of as a *witness* that "A is true", or a *proof of A*. For instance, given $x, y : A$, there will be a type $x =_A y$ that will have elements only if "x and y are equal". More generally, we will see many *type-formers*, i.e. ways to build new types based on existing ones, and rules to build elements of these types (for instance function types $A \to B$, product types $A \times B$, coproduct types $A \sqcup B$, and many others), and each of these can be interpreted as "things with elements" or as "propositions". A summary of analogies between type formers and propositional connectives can be found in table 4.1.

There is also a stricter notion of equality, called *judgmental equality*, written $x \equiv y : A$. This equality statement also only makes sense when we already know that x and y are of type A. When $x \equiv y : A$ (abbreviated $x \equiv y$), we say that "x and y are judgmentally equal" or that they are "equal by definition": meta-theoretically it corresponds to a syntactic equivalence between terms of the theory, typically done in the background in a proof assistant. For instance if we later define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) :\equiv x + x$, then we can substitute x for any natural number and still have the same equality by definition, e.g. $f(3) \equiv 3 + 3$.

---

[1] Here $\mathcal{U}$ denotes a universe of types, defined in section 4.2.2.

For readers interested in learning more about (homotopy) type theory, we greatly recommend the *Homotopy Type Theory* book [Uni13], especially its introductory chapters 1 and 2. For a treatment of the theory as formal system, with judgments and deduction rules, see appendix A of the same book.

4.2 Basic type formers

We start by describing the basic rules to build types and their elements in type theory.

4.2.1 Function types

Given types A and B, we can define the *type $A \to B$ of functions* from A to B, whose elements are called *functions* or *maps*. An element $f : A \to B$ is given by an expression $\lambda x.\phi$ where ϕ is an expression such that $\phi : B$ is derivable from the assumption $x : A$. This can also be written $f(x) :\equiv \phi$.

Given a term $a : A$, we have a term $f(a) : B$ with the computation rule $f(a) \equiv \phi[a/x]$ (called β-reduction), where the expression $\phi[a/x]$ denotes the expression ϕ where every occurrence of x has been replaced by the term a. We also assume the η-expansion rule stating that given $f : A \to B$, we have $f \equiv \lambda x.f(x)$.

When A and B are thought of as propositions, the type $A \to B$ is thought of as the proposition "A implies B", since a proof of $A \to B$ is a function that maps any proof of A to a proof of B.

4.2.2 Universes

In set theory, we can talk about families of sets $(X_i)_{i \in I}$ indexed by some set I. To do something similar with types, we need something like a "type of all types" $\mathcal{U}$, so a family of types indexed by a type I can be defined as a map $X : I \to \mathcal{U}$. Due to the same kind of paradoxes one encounters when trying to define a "set of all sets" in set theory, there cannot be such a "type of all types". In set theory, those paradoxes can be circumvented by axiomatically postulating in hierarchy of *Grothendieck universes*. We do something similar in type theory by postulating the existence of types $\mathcal{U}_k$ called *universes*, whose elements $X : \mathcal{U}_k$ are themselves types, and such that every universe is an element of some bigger universe $\mathcal{U}_k : \mathcal{U}_{k+1}$.[1] We will often deal with only one universe, in which case we omit the index and simply write $\mathcal{U}$.

We ask for all our universes to be closed under the previously defined type formation laws, i.e. given $A, B : \mathcal{U}$, we have

$$A \to B : \mathcal{U}$$
$$A \times B : \mathcal{U}$$
$$A \sqcup B : \mathcal{U}$$

Equipped with universes, we can define a *type family* to be a map $B : A \to \mathcal{U}$ for B a type and $\mathcal{U}$ a universe. We also say that B is a family of types indexed by A.

Given a type B, we may also write $B : A \to \mathcal{U}$ for the *constant type family* $B(x) :\equiv B$.

If every type $B(x)$ for $x : A$ is thought of as a proposition, then the family B can be thought of as a *predicate* over A.

---

[1] The index $k + 1$ in the notation $\mathcal{U}_{k+1}$ is purely meta-theoretic: there is no such a thing as a "family of all universes" indexed by the type of natural numbers $\mathbb{N}$ in type theory.

4.2.3 Dependent functions

Let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ be a family of types. We can define a type

$$\prod_{x:A} B(x) : \mathcal{U}$$

called the *type of dependent functions* (or dependent maps) from $x : A$ to $B(x)$ (compare with the previous *type of functions* from A to B). The type $\prod_{x:A} B(x)$ is also called the dependent product of the family B over A. As the name suggests, elements of $\prod_{x:A} B(x)$ are given by expressions $\lambda x.\phi$, where ϕ is a term such that $\phi : B(x)$ is derivable from the assumption $x : A$. As before, we may write $f(x) :\equiv \phi$ instead of $f :\equiv \lambda x.\phi$.

Given a term $a : A$, we can build a term $f(a) : B(a)$, with the same computation rule as for functions

$$f(a) \equiv \phi[a/x].$$

We also assume the same η-expansion rule $f \equiv \lambda x.f(x)$.

If B is thought of as a predicate over A, then the type $\prod_{x:A} B(x)$ can be thought of as the proposition "$\forall x : A, B(x)$", as an element $f : \prod_{x:A} B(x)$ associates to every element $x : A$ a proof of $B(x)$.

When B is a constant type family, then the type $\prod_{x:A} B(x)$ behaves exactly like the type $A \to B$. Because of this we will behave as if those were the same type, i.e.

$$(A \to B) \equiv (\prod_{x:A} B).$$

Remark 4.2.1. When considering iterated function types, for instance given $f : \prod_{x:A} \prod_{y:B(x)} (C(x) \to D(y))$, we will often abuse notation by writing $f(x, y, z)$ instead of $f(x)(y)(z)$.

4.3 Inductive types

Armed with universes and dependent functions, we can now introduce more types and type formers through the general framework of inductive type definitions. *Inductive types* are defined in a three-step process:

1. **introduction rules:** how to create elements of that type,

2. **elimination rules:** how to build (dependent) maps from this type to another type,

3. and finally, **computation rules** relating the introduction and elimination maps.

For the general theory of inductive types in homotopy type theory, see chapter 5 of [Uni13].

4.3.1 Product types

Given types A and B, we can define the *product type* $A \times B$.

1. **introduction:** given $x : A$ and $y : B$, we can build a term $(x, y) : A \times B$,

2. **elimination:** given a type family $C : (A \times B) \to \mathcal{U}$ and a dependent map $f : \prod_{x:A} \prod_{x:B} C((x, y))$, one can define a map

$$\text{uncurry}(f) : \prod_{u:A \times B} C(u),$$

3. **computation:** for every $x : A, y : B$, the map uncurry (f) satisfies

$$\text{uncurry}\,(f)((x, y)) \equiv f(x)(y)$$

The introduction principle says that any pair (x, y) determines an element of $A \times B$, while the elimination principle encodes in a way the fact that every element of $A \times B$ is of this form.

Using uncurry, we can define maps $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, with computation rules

$$\pi_1((x, y)) \equiv x$$
$$\pi_2((x, y)) \equiv y$$

When A and B are thought of as propositions, the type $A \times B$ corresponds to the proposition "A and B", since a proof of "A and B" consists of a pair of a proof of A and a proof of B.

Combining function types and product types we can define logical equivalences between types:

Definition 4.3.1. The type $A \leftrightarrow B$ of *logical equivalences* between types A and B is defined to be

$$A \leftrightarrow B \quad :\equiv \quad (A \to B) \times (B \to A).$$

Just like in [Uni13], we use the term logical equivalence to distinguish with the stronger notion of equivalence of types to be defined in definition 4.5.1. The notion of logical equivalence is only concerned with the interpretation of types as propositions, not distinguishing for instance the type of natural numbers and the type of booleans, as both are non-empty.

4.3.2 Dependent sums

Let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ be a family of types. We can define a type $\sum_{x:A} B(x)$ called the *dependent sum* of B over A, also called type of dependent pairs or the total space of the family B (when thinking of types as "spaces" in the homotopical sense).

1. **introduction:** given $x : A$ and $y : B(x)$, we can build an element $(x, y) : \sum_{x:A} B(x)$,

2. **elimination:** given a type family $C : (\sum_{x:A} B(x)) \to \mathcal{U}$ and a dependent map $f : \prod_{x:A} \prod_{y:B(x)} C((x, y))$, one can define a map
$$\text{uncurry}\,(f) : \prod_{u:\,\sum_{x:A} B(x)} C(u)$$
just like for product types,

3. **computation:** as before, for every $x : A, y : B(x)$, the map uncurry (f) satisfies
$$\text{uncurry}\,(f)((x, y)) \equiv f(x)(y)$$

Just like for product types, we can define projection maps

$$\pi_1 : \sum_{x:A} B(x) \to A$$
$$\pi_2 : \prod_{p:\,\sum_{x:A} B(x)} B(\pi_1(p))$$

that satisfy

$$\pi_1((x, y)) \equiv x$$
$$\pi_2((x, y)) \equiv y$$

Notation 4.3.2. We will sometimes write ΣAB as a shorthand for $\sum_{x:A} B(x)$.

If B is thought of as a predicate over A, then the type $\sum_{x:A} B(x)$ can be thought of as the proposition "$\exists x : A, B(x)$", as an element $f : \sum_{x:A} B(x)$ consists of a pair (x, y) where $x : A$ is the "witness of existence" and $y : B(x)$ is a proof that x satisfies the predicate B.

Just as dependent products types collapse to function types when the family B is constant, dependent sum types collapse to product types when B is constant, and we will use this identification as if it were a judgmental equality $A \times B \equiv \sum_{x:A} B$.

4.3.3 Coproduct types

Given types A and B, we can define the *coproduct type* (or disjoint union) $A \sqcup B$ and *injection maps* $\iota_1 : A \to A \sqcup B$ and $\iota_2 : B \to A \sqcup B$. Given a type family $C : A \sqcup B \to \mathcal{U}$ and maps $f : \prod_{x:A} C(\iota_1(x)), g : \prod_{y:B} C(\iota_2(y))$, we can define a map $[f, g] : \prod_{z:A \sqcup B} C(z)$, the *coproduct elimination map*, with computation rules

$$[f, g](\iota_1(x)) \equiv f(x)$$
$$[f, g](\iota_2(y)) \equiv g(y)$$

for $x : A, y : B$.

From a logical point of view, the type $A \sqcup B$ corresponds to the proposition "A or B", as any proof of A gives a proof of $A \sqcup B$ through ι_1 and similarly for B and ι_2. The elimination map for coproducts corresponds to the logical principle that $A \vee B \implies C$ as soon as both $A \implies C$ and $B \implies C$.

4.3.4 Unit type

There is a type $1 : \mathcal{U}$ called the *unit type* with an element $\star : 1$ such that for every type family $A : 1 \to \mathcal{U}$ and element $a : A(\star)$, there is a map $\mathrm{ind}_1(A, a) : \prod_{x:1} A(x)$ with computation rule $\mathrm{ind}_1(A, a, \star) \equiv a$.

Logically, the unit type corresponds to the true proposition, since it has a proof $\star : 1$ and for every type A, one can prove "$A \implies 1$" (i.e. one can find a map $f : A \to 1$, defined by $f(x) :\equiv \star$).

4.3.5 Empty type

There is a type $\varnothing : \mathcal{U}$ called the *empty type* with *no element*, i.e. no constructor. Its elimination principle in turn says that given a type family $A : \varnothing \to \mathcal{U}$, there always exists a map $\mathrm{ind}_\varnothing(A) : \prod_{x:\varnothing} A(x)$.

This type corresponds to the false proposition, as there is no way to construct a proof of it (i.e. an element of it). From this logical perspective, its elimination principle corresponds to the *principle of explosion* stating that a contradiction implies every proposition.

4.3.6 Natural numbers

There is a type $\mathbb{N} : \mathcal{U}$ called the *type of natural numbers*, with two constructors

$$0 : \mathbb{N} \tag{4.1}$$
$$\mathrm{suc} : \mathbb{N} \to \mathbb{N} \tag{4.2}$$

Its elimination principle states that given a type family $A : \mathbb{N} \to \mathcal{U}$ and

$$a_0 : A(0)$$
$$a_{\text{suc}} : \prod_{n:\mathbb{N}} (A(n) \to A(\text{suc}\,(n)))$$

one can build a function

$$\text{ind}_{\mathbb{N}}(A, a_0, a_{\text{suc}}) : \prod_{n:\mathbb{N}} A(n)$$

such that

$$\text{ind}_{\mathbb{N}}(A, a_0, a_{\text{suc}}, 0) \equiv a_0$$
$$\text{ind}_{\mathbb{N}}(A, a_0, a_{\text{suc}}, suc(n)) \equiv a_{\text{suc}}(n, \text{ind}_{\mathbb{N}}(A, a_0, a_{\text{suc}}, n)) \text{ (for } n : \mathbb{N})$$

This elimination principle encompasses both the definition of functions $\mathbb{N} \to A$ by recursion and the proof of propositions "$\forall n : \mathbb{N}, P(n)$" by induction.

4.4 Identity types and the homotopical interpretation of types

As promised in the introduction of this chapter, we need to introduce, for every type $A : \mathcal{U}$, a type family

$$(- =_A -) : A \to A \to \mathcal{U}$$

such that elements of $x =_A y$ correspond to "witnesses that x and y are equal". This type family is also defined by an induction principle, but this induction principle characterizes the entire family, not just the individual types $x =_A y$.

Definition 4.4.1. Let $x : A$. We define a type family $(x =_A -) : A \to \mathcal{U}$ by induction, with a constructor

$$\text{refl}_x : x =_A x$$

called *reflexivity* and an elimination principle stating that given any family $C : \prod_{y:A} (x =_A y \to \mathcal{U})$ and element $c_{\text{refl}} : C(x, \text{refl}_x)$, one can build a term

$$\text{ind}_=(C, c_{\text{refl}}) : \prod_{y:A} \prod_{p:x=_Ay} C(y, p)$$

such that

$$\text{ind}_=(C, c_{\text{refl}}, x, \text{refl}_x) \equiv c_{\text{refl}}$$

The type $x =_A y$ is called the *identity type* or equality type between x and y. Given a witness $p : x =_A y$, we say that x and y are equal or that they are *typally equal* (to explicitly distinguish with judgmental equality). When the type A is clear from the context, we may simply write $x = y$ instead of $x =_A y$.

Definition 4.4.2. We write $x \neq y$ as a shorthand for $\neg(x = y)$, i.e. $(x = y) \to \emptyset$, where $\neg X :\equiv X \to \emptyset$.

In ordinary Martin Löf type theory, the equality type is typically interpreted only through its propositional content: "$x = y$ is inhabited if x and y are equal". This is quite reasonable when thinking of types as sets: in a set, equality is a proposition, a first-order statement, and there is no meaning behind questions such as "*how* are x and y equal ?"

In their seminal paper [HS98], Martin Hofmann and Thomas Streicher showed it is possible to give an interpretation of MLTT such that types are interpreted as groupoids, and such that equality types $x =_A y$ are interpreted as the (discrete) groupoid of isomorphisms between x and y in A, showing that the following statement (*uniqueness of identity proofs*) is **not** derivable in MLTT

$$\prod_{A:\mathcal{U}} \prod_{x:A} \prod_{y:A} \prod_{p:x=y} \prod_{q:x=y} p = q.$$

However there is no way in plain MLTT to construct a type A and elements $x, y : A$ such that the type $x =_A y$ contains two different elements. In other words uniqueness of identity proofs is *independent* from the rules of type theory.

Homotopy type theory adds axioms and rules to Martin Löf's theory so that (among other things) interesting examples of such types can be constructed. Actually, in HoTT, even iterated identity types can be non-trivial, so that types should be thought of as ∞-groupoids rather than just sets or groupoids. Because of this interpretation, thinking of types as kinds of "spaces" or ∞-groupoids, we also use the following terminology:

Definition 4.4.3. Given a type A, the elements of A are called *points*. Given two points $x, y : A$, the type $x =_A y$ is called the *path type* between x and y in A. An element $p : x =_A y$ is called a path from x to y, and the reflexivity identity refl_x is called the *constant path* at x.

The induction principle

$$\text{ind}_=(C, c_{\text{refl}}) : C(x, \text{refl}_x) \to \prod_{y:A} \prod_{p:x=_A y} C(y, p)$$

is called path induction.

Before adding new axioms to get examples of non-trivial identity types, we can already say a lot about the higher groupoidal structure of those types using just path induction.

Definition 4.4.4. Let A be a type, $x, y, z : A$, $p : x = y$ and $q : y = z$. The *concatenation* (or composition) of p and q is a path $p \cdot q : x = z$. The map

$$(- \cdot -) : x = y \to y = z \to x = z$$

is defined by path induction on the first argument, and we take by definition

$$\text{refl}_x \cdot q :\equiv q.$$

Thinking of paths as simply witnesses of equality, concatenation witnesses the *transitivity* of equality.

Fact 4.4.5. *We can prove that $p \cdot \text{refl}_y = p$ by path induction on p.*

Remark 4.4.6. With this definition, path concatenation is judgmentally unital on the left and typally unital on the right. We could have defined concatenation by path induction on the right argument instead to reverse which side is judgmentally unital, or do path induction twice and only get a judgmental equality $\text{refl} \cdot \text{refl} \equiv \text{refl}$. It turns out these three possible definitions of concatenation are typally equal (and this is once again shown by path induction): what really matters is that unitality of concatenation holds typally.

Fact 4.4.7. *Given $x, y, z, t : A, p : x = y, q : y = z, r : z = t$, we have $(p \cdot q) \cdot r = p \cdot (q \cdot r)$.*

The proof being, again, by path induction.

Definition 4.4.8. Let A be a type, and $x, y : A$ and $p : x = y$ a path from x to y in A. We define a path $p^{-1} : y = x$ in A by path induction such that $\mathrm{refl}\,_x^{-1} :\equiv \mathrm{refl}\,_x$.

Fact 4.4.9. *Let $p : x = y$ be a path in a type A. We have $p \cdot p^{-1} = \mathrm{refl}\,_x$ and $p^{-1} \cdot p = \mathrm{refl}\,_y$. The proof is once again by path induction on p.*

The previous definitions and facts show that the identity types equip any type A with a structure of groupoid : equalities can be composed and reversed in an associative and unital way. This associativity and unitality themselves hold up to equality, i.e. there are *paths between paths* witnessing those laws. These *higher paths* themselves can be composed in multiple ways, give rise to a 2-groupoidal structure on types. Even higher-dimensional operations on paths can be defined, always by path induction, resulting in all the higher laws and coherences to be expected from ∞-groupoids.

Remark 4.4.10. This statement can be made formal by proving that the usual homotopy types (as defined in set theory) define a set-theoretical model homotopy type theory [KL21].

In particular given a point $a : A$, the type $a =_A a$ carries the structure of a group (or rather, an ∞-group).

Definition 4.4.11. A *pointed type* is a pair (A, a) where A is a type and $a : A$. Given a universe $\mathcal{U}$, the type of pointed types in $\mathcal{U}$ is by definition $\mathcal{U}_\bullet :\equiv \sum_{A : \mathcal{U}} A$.

Definition 4.4.12. Given a type A and a point $a : A$, the type $a =_A a$ is denoted $\Omega(A, a)$ and called the *loop space* of A. The loop space of A is canonically pointed at $\mathrm{refl}\,_a : a =_A a$, so Ω defines a function $\Omega : \mathcal{U}_\bullet \to \mathcal{U}_\bullet$.

The loop space operation can thus be iterated, and yields for every pointed type (A, a) a family of "higher groups" $\Omega^n(A, a)$ that contain a lot of information on the "topological" or rather "homotopical" structure of the type A. We'll get into more detail about this in section 4.12 on how to define homotopy groups in HoTT.

Now that we understand why types behave like ∞-groupoids, we can see that functions behave functorially with respect to this structure.

Fact 4.4.13. *Given a map $f : A \to B$ and $x, y : A$, there is a canonical map*

$$\mathrm{ap}_f : x =_A y \to f(x) =_B f(y)$$

called the action of f on paths, *defined by path induction such that $\mathrm{ap}_f\,(\mathrm{refl}\,_x) \equiv \mathrm{refl}\,_{f(x)}$.*

In set theory, the implication $x = y \implies f(x) = f(y)$ simply means that every function is a *congruence* with respect to equality. But here in homotopy type theory, the types $x =_A y$ and $f(x) =_B f(y)$ may contain non-trivial elements, and the action of f on equality types thought of as path types becomes a function of its own. This action on paths is moreover compatible with the (higher) groupoid laws.

Fact 4.4.14. *Let $f : A \to B$ be any map. We have:*

1. $\mathrm{ap}_f\,(\mathrm{refl}\,_x) = \mathrm{refl}\,_{f(x)}$

2. $\mathrm{ap}_f\,(p \cdot q) = \mathrm{ap}_f\,(p) \cdot \mathrm{ap}_f\,(q)$

3. $\mathrm{ap}_f\,(p^{-1}) = (\mathrm{ap}_f\,(p))^{-1}$

And similarly for higher definable compositions between higher paths.

This "automatic ∞-functoriality" of every map follows yet again by path induction: this really shows the power of interpreting equality types as types of paths.

Let's now consider a dependent map $f : \prod_{x:A} B(x)$. Given $x, y : A$, we might want a similar map $(x = y) \to (f(x) = f(y))$. However in this case we have $f(x) : B(x)$ and $f(y) : B(y)$, and a priori we don't know that $B(x) \equiv B(y)$, so there is no meaning behind the expression $f(x) = f(y)$, as $f(x)$ and $f(y)$ are not elements of the same type. We could try to add a rule to the type theory to derive $B(x) \equiv B(y)$ whenever we can construct some $p : x = y$, but it turns out this would render path types trivial (as well as making judgmental equality undecidable).

The more reasonable approach is to use a path $p : x = y$ to compare the types $B(x)$ and $B(y)$, so that the elements $f(x)$ and $f(y)$ can then themselves be compared.

Definition 4.4.15. Let $A : \mathcal{U}$ be a type and $B : A \to \mathcal{U}$ a type family over A. For every $x, y : A$ and $p : x = y$, there is a map

$$p_* : B(x) \to B(y)$$

called *transport over p*, defined by path induction so that

$$(\mathrm{refl}\,_x)_* \equiv \mathrm{id}_{B(x)}.$$

Following the notation of [Uni13], we write

$$\mathrm{transport}\,_p^B(-) : B(x) \to B(y)$$

for the map p_* when the family B needs to be made explicit.

Now given a path $p : x = y$ and a dependent map $f : \prod_{x:A} B(x)$ as above, we can transport $f(x)$ through p to get an element $p_*(f(x)) : B(y)$, which can then be compare with $f(y)$.

Fact 4.4.16. *Let $A : \mathcal{U}$, $B : A \to \mathcal{U}$, $f : \prod_{x:A} B(x)$, and let $x, y : A$. There is a map*

$$\mathrm{apd}_f : \prod_{p:x=y} \left(p_*(f(x)) =_{B(y)} f(y) \right)$$

called the dependent action on paths, defined by path induction so that $\mathrm{apd}_f(\mathrm{refl}\,_x) \equiv \mathrm{refl}\,_{f(x)}$.

The following definition allows us to rewrite the type of apd_f in a nicer way.

Definition 4.4.17. Given a type family $B : A \to \mathcal{U}$, elements $x, y : A$, a path $p : x = y$ and elements $u : B(x)$ and $v : B(y)$, the type $p_*(u) =_{B(y)} v$ is written

$$u =_B^p v$$

or simply

$$u =^p v$$

and called the *type of paths from u to v over p* in the family B, or the type of dependent paths from u to v over p in B.

Hence we have

$$\mathrm{apd}_f : \prod_{p:x=y} f(x) =^p f(y).$$

This notion of path type over p will become more intuitive as we study path types between dependent pairs and dependent maps in the next section.

For now we note that the dependent action on paths applied to non-dependent functions is closely related to the non-dependent action on paths.

Fact 4.4.18 ([Uni13, Lem 2.3.5])**.** *Let A and B be types, and see B as a constant type family over A. Let $x, y : A$, $p : x = y$ and $u : B$. Then there is a path*

$$\text{transportconst}_p^B(u) : p_*(u) =_B u.$$

In other words, transport in constant type family is equal to the identity.

Let $f : A \to B$ be a map between types, $x, y : A$ and $p : x = y$. Seeing B as a constant type family over A, we can compute both $\text{ap}_f(p) : f(x) = Bf(y)$ and $\text{apd}_f(p) : p_*(f(x)) =_B f(y)$.

Fact 4.4.19 ([Uni13, Lem 2.3.8])**.** *We have*

$$\text{apd}_f(p) = \text{transportconst}_p^B(f(x)) \cdot \text{ap}_f(p)$$

Diagrammatically, we have a commutative triangle of paths

$$
\begin{array}{ccc}
p_*(f(x)) & \xrightarrow{\text{apd}_f(p)} & f(y) \\
{\scriptstyle\text{transportconst}_p^B(f(x))}\Big\downarrow & \nearrow & \\
f(x) & {\scriptstyle\text{ap}_f(p)} &
\end{array}
$$

In particular, composing with $\text{transportconst}_p^B(f(x))$ or its inverse, we can go back and forth between the types $f(x) = f(y)$ and $p_*(f(x)) = f(y)$ in a mutually inverse way: these types are equivalent, in a sense we define in the next section (definition 4.5.1).

4.5 Characterizing identity types

In the previous section, we have seen what kind of structure is carried by the identity types $x =_A y$ in homotopy type theory, but we have yet to actually prove that two things are equal. In the case where A is an inductive type, we can actually use its induction principle to "characterize" equality in A.

Here by "characterize", we informally mean finding for every $x, y : A$ a type $B(x, y)$ that is "equivalent" to $(x =_A y)$ and also a "simpler expression of x and y". For that to make sense we first need to define what it means for two types to be equivalent.

Definition 4.5.1. Let $f : A \to B$ be a map between types. We say f is an *equivalence* if

- it has a left inverse g, i.e. $g : B \to A$ such that $\prod_{x:A} g(f(x)) = x$

- it has a right inverse h, i.e. $h : B \to A$ such that $\prod_{y:B} f(h(y)) = y$

In other words, we define a type $\text{isEquiv}(f)$ as

$$\text{isEquiv}(f) :\equiv \Big(\sum_{g:B \to A} \prod_{x:A} g(f(x)) = x \Big) \times \Big(\sum_{h:B \to A} \prod_{y:B} f(h(y)) = y \Big)$$

An *equivalence* is a pair (f, t) where $f : A \to B$ and $t : \text{isEquiv}(f)$. We write $A \simeq B$ for the type of equivalences from A to B, i.e.

$$(A \simeq B) :\equiv \sum_{f:A \to B} \text{isEquiv}(f)$$

Two types A and B are said to be equivalent if there is an equivalence $(f, t) : A \simeq B$.

Remark 4.5.2. In practice, we will often write $f : A \simeq B$ instead of $(f, t) : A \simeq B$, making the proof $t : \text{isEquiv}(f)$ implicit, as will be justified by fact 4.7.11 and example 4.7.14.

Remark 4.5.3. We will also refer to either the left or right inverse of f given by t as "the" *inverse* of f since function extensionality will imply that they are a posteriori equal (fact 4.6.5).

One may wonder why we define equivalences in this ways: why not ask for a single map $g : B \to A$ that is both a left inverse and a right inverse ? As indeed, in definition 4.5.1, the maps g and h will necessarily satisfy $\prod_{y:B} g(y) = h(y)$. Nonetheless, while this would give a *logically equivalent* (definition 4.3.1) definition of equivalence, it would not define an equivalent type in the above sense (see [Uni13, Sec 4.1]).

Remark 4.5.4. There are actually many equivalent ways to define the notion of equivalence, so much so that the original homotopy type theory book [Uni13] has an entire chapter dedicated to comparing them.

We can now state a first characterization result.

Fact 4.5.5 ([Uni13, Thm 2.6.2]). *Let A, B be types, and $u, v : A \times B$. There is an equivalence*

$$(u = v) \simeq \big((\pi_1(u) = \pi_1(v)) \times (\pi_2(u) = \pi_2(v))\big).$$

Proof idea. The map from left to right is given by $p \mapsto (\text{ap}_{\pi_1}(p), \text{ap}_{\pi_2}(p))$. Going from right to left, we can use the uncurry elimination principle for product types to reduce to the case where $u :\equiv (x, y)$ and $v :\equiv (x', y')$ with $x, x' : A$ and $y, y' : B$. We can then use uncurry again to reduce to showing the following:

$$\prod_{x:A} \prod_{x':A} \prod_{p:x=x'} \prod_{y:B} \prod_{y':B} \prod_{q:y=y'} (x, y) = (x', y')$$

By path induction on p and q, this reduces to proving that

$$\prod_{x:A} \prod_{y:B} (x, y) = (x, y)$$

which is true by reflexivity.

That the two maps we just built are indeed inverse to one another results from more path inductions and applications of uncurry. $\qquad\square$

An analogous statement also holds for dependent sums, with the added subtlety that we need to make use of paths over paths.

Fact 4.5.6 ([Uni13, Thm 2.7.2]). *Let A be a type, $B : A \to \mathcal{U}$ a type family and $u, v : \sum_{x:A} B(x)$. There is an equivalence*

$$(u = v) \simeq \sum_{p:\pi_1(u)=\pi_1(v)} \pi_2(u) =^p \pi_2(v)$$

To be able to use these results in practice, we will also need to characterize the types of dependent paths $x =^p y$, which we recall is by definition $p_*(x) = y$, so we need to understand how transport computes. An important family of examples is given by the path types themselves:

Fact 4.5.7 ([Uni13, Lem 2.11.2]). *Let A be a type, $x, y, z : A$, $p : x = y$ and $q : y = z$. We have*

$$\text{transport}_p^{(x=-)}(q) = p \cdot q.$$

Proof. By path induction on p, we are left with proving that $q = \text{refl} \cdot q$, with is true by remark 4.4.6.
$\square$

Back to characterizing path types in inductive types, we have:

Fact 4.5.8. *Given $x, y : 1$, there is an equivalence*

$$(x = y) \simeq 1$$

For coproducts and natural numbers, the type $x = y$ depends on the way x and y are constructed, so we have to define our characterization by induction.

Fact 4.5.9 ([Uni13, Sec 2.12]). *Let A, B be types. Define a family of types*

$$\text{code}_{A \sqcup B} : A \sqcup B \to A \sqcup B \to \mathcal{U}$$

by

$$\text{code}_{A \sqcup B}(\iota_1(x), \iota_1(x')) \simeq x =_A x'$$
$$\text{code}_{A \sqcup B}(\iota_2(y), \iota_2(y')) \simeq y =_B y'$$
$$\text{code}_{A \sqcup B}(\iota_1(x), \iota_2(y)) \simeq \varnothing$$

Then for every $u, v : A \sqcup B$, there is an equivalence

$$(u = v) \simeq \text{code}_{A \sqcup B}(u, v).$$

Fact 4.5.10 ([Uni13, Sec 2.13]). *Define*

$$\text{code}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \to \mathcal{U}$$

by

$$\text{code}_{\mathbb{N}}(0, 0) :\equiv 1$$
$$\text{code}_{\mathbb{N}}(\text{suc}(m), 0) :\equiv \varnothing$$
$$\text{code}_{\mathbb{N}}(0, \text{suc}(n)) :\equiv \varnothing$$
$$\text{code}_{\mathbb{N}}(\text{suc}(m), \text{suc}(n)) :\equiv \text{code}_{\mathbb{N}}(m, n)$$

Then, for every $m, n : \mathbb{N}$, there is an equivalence

$$(m = n) \simeq \text{code}_{\mathbb{N}}(m, n)$$

Characterizations of identity types can also be transported through equivalences:

Fact 4.5.11. *Let $f : A \simeq B$ be an equivalence of types. Then for every $x, y : A$, the map*

$$\text{ap}_f : x = y \to f(x) = f(y)$$

is also an equivalence.

Let $A, B : \mathcal{U}$ and $f, g : A \to B$. It is tempting to write that

$$(f = g) \simeq \prod_{x:A} f(x) = g(x),$$

but this is actually not provable in MLTT. We can show by path induction that there is a canonical map $(f = g) \to \prod_{x:A} f(x) =_B g(x)$, but when trying to build an inverse, we get stuck as there is no general induction principle for function types $A \to B$ like there is for inductive types. The same problem happens with dependent functions, where we can use path induction to build a map

$$(f =_{\prod_{x:A} B(x)} g) \to \prod_{x:A} (f(x) =_{B(x)} g(x))$$

The principle saying that the above map is an equivalence is called function extensionality. In classical dependent type theory it is usually taken as an axiom, but in homotopy type theory it turns out to be implied by the univalence axiom (fact 4.6.4), which is the subject of the next section.

4.6   Univalence

The last type whose identity types we haven't characterized is the universe $\mathcal{U}$. Like for function types, there is no way to give a general characterization of equality between types in MLTT. The role of the univalence axiom is to do precisely this: given a definite answer to the question "what is the type $A =_{\mathcal{U}} B$", and this answer is $A \simeq B$.

Definition 4.6.1. Given types $A, B : \mathcal{U}$, write

$$\text{idtoequiv} : A =_{\mathcal{U}} B \to A \simeq B$$

for the map defined by path induction so that $\text{idtoequiv}\,(\text{refl}\,_A) \equiv \text{id}_A$.

Axiom 4.6.2 (*Univalence axiom*)**.** The map idtoequiv is an equivalence. In particular we have

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B)$$

We write ua for the inverse map

$$\text{ua} : (A \simeq B) \to (A =_{\mathcal{U}} B).$$

Example 4.6.3. Write Bool for the inductive type generated by two constructors false : Bool and true : Bool. There are two equivalences Bool $\simeq$ Bool: the identity id_{Bool} and the map swap : Bool $\to$ Bool that maps true to false and false to true. Hence the map

$$\begin{aligned}
\text{ev}_{\text{true}} : (\text{Bool} \simeq \text{Bool}) &\to \text{Bool} \\
(f, \dots) &\mapsto f(\text{true})
\end{aligned}$$

is an equivalence, with inverse given by

$$\begin{aligned}
\text{Bool} &\to (\text{Bool} \simeq \text{Bool}) \\
\text{true} &\mapsto \text{id}_{\text{Bool}} \\
\text{false} &\mapsto \begin{cases} \text{true} \mapsto \text{false} \\ \text{false} \mapsto \text{true} \end{cases}
\end{aligned}$$

(where the proofs that the resulting maps Bool $\to$ Bool are equivalences being omitted). Through univalence, this means that there are two different paths (up to higher paths) from Bool to Bool in the universe $\mathcal{U}$, given by ua $(\mathrm{id}_{\mathrm{Bool}})$ and ua (swap). Univalence also tells us that ua $(\mathrm{id}_{\mathrm{Bool}}) = \mathrm{refl}_{\mathrm{Bool}}$.

We can explicitly prove that refl $_{\mathrm{Bool}}$ and ua (swap) are different, in the sense of definition 4.4.2, as follows. Write θ for the composite map

$$(\mathrm{Bool} =_{\mathcal{U}} \mathrm{Bool}) \xrightarrow{\mathrm{idtoequiv}} (\mathrm{Bool} \simeq \mathrm{Bool}) \xrightarrow{\mathrm{ev}_{\mathrm{true}}} \mathrm{Bool} \xrightarrow{\psi} \mathcal{U}$$

where the last map $\psi : \mathrm{Bool} \to \mathcal{U}$ is defined by induction so that

$$\psi(\mathrm{false}) :\equiv \varnothing$$
$$\psi(\mathrm{true}) :\equiv 1.$$

Suppose $p : \mathrm{refl}_{\mathrm{Bool}} = \mathrm{ua}\,(\mathrm{swap})$. We have

$$\mathrm{ap}_{\theta}\,(p) : \theta(\mathrm{refl}_{\mathrm{Bool}}) = \theta(\mathrm{ua}\,(\mathrm{swap})).$$

Since $\theta(\mathrm{refl}_{\mathrm{Bool}}) = 1$ and $\theta(\mathrm{ua}\,(\mathrm{swap})) = \varnothing$, this means we can find $p' : 1 = \varnothing$, and hence by transport, a map $1 \to \varnothing$, which we can apply to $\star : 1$ to get an element of $\varnothing$.

The following corollary is far from obvious, and relies crucially on the η-extensionality rule for dependent functions $f \equiv \lambda x.f(x)$.

Fact 4.6.4 ([Uni13, Section 4.9]). *The univalence axiom implies* function extensionality, *i.e. that for any pair of dependent maps $f, g : \prod_{x:A} B(x)$, the canonical map*

$$(f = g) \to \prod_{x:A} (f(x) = g(x))$$

has an inverse

$$\mathrm{funext} : \Big(\prod_{x:A} (f(x) = g(x)) \Big) \to (f = g).$$

Fact 4.6.5. *Given a map $f : A \to B$ and $((g, \alpha), (h, \beta)) : \mathrm{isEquiv}\,(f)$, we have $g = h$.*

Proof. Indeed, given $y : B$ we have

$$\begin{aligned} g(y) &= g(f(h(y))) & (\text{by } \mathrm{ap}_g\,(\beta_y)) \\ &= h(y) & (\text{by } \alpha_{h(y)}) \end{aligned}$$

$\square$

This justifies calling either g or h "the" inverse of f (remark 4.5.3).

4.7   n-Types

We mentioned in section 2.2 the notion of homotopy n-type as homotopy types whose homotopy groups π_k vanish for $k > n$. We could try to give a similar definition for types in HoTT, however we haven't yet defined homotopy groups in this setting. The "higher groups" $\Omega^n(X, x)$ look like homotopy groups, but they are not quite the same thing: $\Omega^n(X, x)$ is a general type, while we want the homotopy group $\pi_n(X, x)$ to be a set, i.e. a 0-type.

The approach taken in [Uni13] is to rather use an inductive definition of n-type. Suppose we managed to define what it means to be a 0-type, then a $(n+1)$-type can be defined to be a type X for which the path types $x = y$ are n-types for every $x, y : X$. It is actually more convenient to start the hierarchy at $n = -2$ with the notion of contractibility.

Let A be a type.

Definition 4.7.1. We write
$$\text{isContr}\,(A) :\equiv \sum_{a:A} \prod_{x:A} a = x$$

and say that A is *contractible* whenever given a witness $t : \text{isContr}\,(A)$. The distinguished element $\pi_1(t) : A$ is called the *center of contraction* of A.

Intuitively, this definition of contractibility could be read "there is a distinguished point $a : A$ such that for every $x : A$, there is a path from a to x". From the point of view of topology, this looks more like a definition of connectedness than a definition of contractibility (to recall section 2.2, a topological space is said to be contractible if it is homotopy equivalent to a singleton space). For instance, a circle seems like it should satisfy the above definition, however that is not the case: indeed this definition is made in homotopy type theory, so the statement

"for every $x : A$, there is a path from a to x"

must be thought to be "continuous" in x. Write $f : \prod_{x:A} a = x$ for a dependent map witnessing the contractibility of A. Given points $x, y : A$, we have paths $f(x) : a = x$, $f(y) : a = y$. Now given a path $p : x = y$, the dependent application of f gives
$$\text{apd}_f(p) : f(x) =^p_{(a=-)} f(y),$$

which should be thought of as a commutative triangle
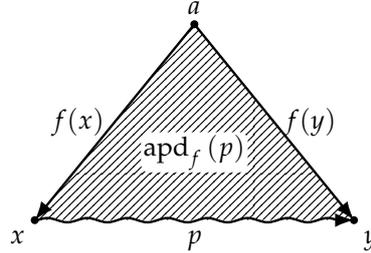
witnessing the equality of paths
$$f(x) \cdot p = f(y)$$

and indeed it can be shown by path induction on p that
$$f(x) =^p_{(a=)-} f(y) \quad \rightarrow \quad f(x) \cdot p = f(y).$$

From this we easily deduce that $p = f(x)^{-1} \cdot f(y)$, and since this is true for every p, we have the following.

Fact 4.7.2. *Let A be a contractible type and $x, y : A$. Then*
$$\prod_{p:x=y} \prod_{q:x=y} p = q$$

and $x = y$ is contractible.

Iterating that fact, we see that a contractible type really has no non-trivial higher paths in any dimension. That contractibility as defined here really corresponds to the topological contractibility introduced in section 2.2 can be formalized as follows.

Fact 4.7.3 ([Uni13, Lem 3.11.3])**.** *A type A is contractible if and only if $A \simeq 1$.*

Hence a contractible type really is a type with "exactly one element", in a homotopically meaningful way. In particular, set-theoretic statements of the form "there exists a unique x such that $P(x)$" would be translated in homotopy type theory as isContr $\left(\sum_{x:X} P(x) \right)$.

An important family of examples of contractible types is given by *singleton types*.

Definition 4.7.4. Let A be a type and $a : A$. The *singleton type* at $a : A$ is the type

$$\sum_{x:A} a = x.$$

Fact 4.7.5 ([Uni13, Lem 3.11.8])**.** *For every type A and $a : A$, the type $\sum_{x:A} a = x$ is contractible.*

Proof. The type $\sum_{x:A} a = x$ is inhabited by the element (a, refl_a). Given another pair (x, p) with $x : A$ and $p : a = x$, an equality between (a, refl_a) and (x, p) corresponds by fact 4.5.6 to a pair of

- an equality $q : x = a$,

- an equality

$$\text{transport}_q^{(a=-)}(\text{refl}_a) = p$$

that last equality corresponds by fact 4.5.7 to an equality $\text{refl}_a \cdot q = p$. Hence we can just take $q :\equiv p$ and conclude by the unitality of concatenation (remark 4.4.6). $\qquad\square$

From contractibility we can inductively define *n*-types.

Definition 4.7.6. Given a type A, we define

$$\text{is-}n\text{-type}(A)$$

by induction on $n \geq -2$:

- is-(-2)-type $(A) :\equiv$ isContr (A),

- is-$(n+1)$-type $(A) :\equiv \prod_{x:A} \prod_{y:A}$ is-n-type $(x = y)$.

Given a witness $t :$ is-n-type (A), we say that A is an *n-type* or that it is of homotopy level n.
We write

$$n\text{-Type} :\equiv \sum_{X:\mathcal{U}} \text{is-}n\text{-type}(X)$$

for the *type of n-types* in $\mathcal{U}$, or the universe of *n*-types.

As a direct corollary of fact 4.7.2, we get that any *n*-type is also an $(n+1)$-type, so the hierarchy of *n*-types is cumulative ([Uni13, Thm 7.1.7]).

Fact 4.7.7 (*Cumulativity* of *n*-types)**.** *Let A be a type and $m \leq n$, then*

$$\text{is-}m\text{-type}(A) \rightarrow \text{is-}n\text{-type}(A),$$

i.e. every m-type is also an n-type whenever $m \leq n$.

Let's now study a few examples.

4.7.1 The case $n = -1$

A type A is a (-1)-type if for every $x, y : A$, the type $x = y$ is contractible. This is the homotopically meaningful way of stating that "A has at most one element". Thinking of types from the point of view of logic, A having "at most one element" means that there is at most one witness to the truth-value of A: if A is true, it is true in a (homotopically) unique way. This is what we expect from propositions in logic: a proposition is either true or it is not, it is not "possibly true in multiple ways" like types are. Hence (-1)-types are called propositions. It turns out that the definition of (-1)-type is equivalent to the following simpler variant.

Definition 4.7.8. We write

$$\mathrm{isProp}\,(A) :\equiv \prod_{x:A} \prod_{y:A} x = y$$

and say that A is a *proposition* whenever given a witness $t : \mathrm{isProp}\,(A)$. We write hProp for the type $\sum_{X:\mathcal{U}} \mathrm{isProp}\,(X)$ of propositions (or *universe of propositions*).

Remark 4.7.9. Propositions are called "mere propositions" in [Uni13] to keep in mind that every type can still be thought of as a proposition, although a proof-relevant one.

Fact 4.7.10. *A type A is a (-1)-type if and only if it is a proposition.*

Proof. Suppose that A is a (-1)-type. Then for every $x, y : A$, $x = y$ is by definition contractible and in particular inhabited, so A is a proposition. More formally, given $t : \mathrm{is\text{-}{-1}\text{-}type}\,(A)$, we have

$$(\lambda x.\lambda y.\pi_1(t(x,y))) : \mathrm{isProp}\,(A)$$

Reciprocally, suppose $t : \mathrm{isProp}\,(A)$, and let $x, y : A$. We have $t(x) : \prod_{z:A} x = z$, so $(x, t(x)) : \mathrm{isContr}\,(A)$, so fact 4.7.2 implies that $\mathrm{isContr}\,(x' = y')$ is contractible for every $x', y' : A$, in particular taking $x' :\equiv x$ and $y' :\equiv y$ yields the desired result. $\square$

The notion of proposition is very useful to represent proof-irrelevant notions. For instance, the sentence "the map f is an equivalence" means that we have a witness $t : \mathrm{isEquiv}\,(f)$. If "being an equivalence" is a proposition, then that means that we don't have to worry about possibly different witnesses of equivalence, and indeed we chose the definition of isEquiv so that this is true (as already mentioned in remark 4.5.2).

Fact 4.7.11 ([Uni13, Thm 4.3.2]). *Let $f : X \to Y$ be a map between types. Then $\mathrm{isEquiv}\,(f)$ is a proposition.*

More generally, every type we wrote as "isSomething(…)" is a proposition:

Fact 4.7.12. *Let A be a type. Then the types*

$$\mathrm{isContr}\,(A)$$
$$\mathrm{is\text{-}}n\text{-}\mathrm{type}\,(A)$$
$$\mathrm{isProp}\,(A)$$

are all propositions.

Propositions are especially useful in dependent sum types, as they greatly simplify the characterization of path types in dependent sums.

Fact 4.7.13. *Let A be a type, and $B : A \to \mathcal{U}$ such that $\prod_{x:A} \mathrm{isProp}\,(B(x))$. Let $u, v : \sum_{x:A} B(x)$. Then we have*

$$(u = v) \simeq (\pi_1(u) \simeq \pi_1(v)).$$

Example 4.7.14. Let $(f, t), (g, u) : X \simeq Y$. Then

$$(f, t) =_{X \simeq Y} (g, u) \quad \simeq \quad f =_{X \to Y} g,$$

justifying once again remark 4.5.2.

Because propositions have at most one element, any logical equivalence between propositions implies that they are equivalent.

Fact 4.7.15. *Let A and B be propositions, and suppose that $A \leftrightarrow B$. Then $A \simeq B$ and any map $A \to B$ or $B \to A$ is an equivalence.*

In particular, the "if and only if" of fact 4.7.10 actually gives an equivalence

$$\mathrm{isProp}\,(A) \simeq \mathrm{is}\text{-}(-1)\text{-type}\,(A)$$

which also implies an equivalence

$$\mathrm{hProp} \simeq (-1)\text{-Type}.$$

4.7.2 The case $n = 0$

A type A is a 0-type if every parallel paths $p, q : x = y$ in A are uniquely equal, i.e. $\mathrm{isContr}\,(x = y)$. This means that A may have multiple elements, but the relation of equality in A is propositional and may not contain higher homotopical information. In classical topology, 0-types correspond to spaces that are homotopy equivalent to discrete spaces, i.e. sets. Since in HoTT everything is already considered "up to homotopy", we simply call these types sets. The definition of sets can be simplified in the same way that the definition of propositions was.

Definition 4.7.16. We write

$$\mathrm{isSet}\,(A) :\equiv \prod_{x:A} \prod_{y:A} \mathrm{isProp}\,(x = y)$$

$$\equiv \prod_{x:A} \prod_{y:A} \prod_{p:x=y} \prod_{q:x=y} p = q$$

and say that A is a *set* whenever given $t : \mathrm{isSet}\,(A)$. We write hSet for the *type of sets* (or *universe of sets*)

$$\mathrm{hSet} :\equiv \sum_{X:\mathcal{U}} \mathrm{isSet}\,(X)$$

Fact 4.7.17. *"Being a set" is logically equivalent to "being a 0-type", in other words for any type A, we have*

$$\mathrm{isSet}\,(A) \leftrightarrow \mathrm{is}\text{-}0\text{-type}\,(A)$$

Moreover, being a set is a proposition :

$$\mathrm{isProp}\,(\mathrm{isSet}\,(A)),$$

so by fact 4.7.15 we actually have

$$\mathrm{isSet}\,(A) \simeq \mathrm{is}\text{-}0\text{-type}\,(A)$$

which also entails that

$$\mathrm{hSet} \simeq 0\text{-Type}.$$

Example 4.7.18. The type $\mathbb{N}$ of natural numbers is a set, as can be seen from the characterization of path types in $\mathbb{N}$ (fact 4.5.10).

4.7.3 The fundamental theorem of identity types

Fact 4.7.5 states that singleton types are contractible. In fact this can be used to as a full characterization of identity types, a characterization so important that it has been coined *the fundamental theorem of identity types* by Egbert Rijke in his introductory book to homotopy type theory [Rij25, Chapter 11].

Fact 4.7.19 (*The fundamental theorem of identity types* [Rij25, Thm 11.2.2]). *Let A be a type, $a : A$ and $B : A \to \mathcal{U}$ a type family and $b : B(a)$, so that by path induction we can define a map*

$$f : \prod_{x:A} (a = x \to B(x))$$

such that $f(a, \mathrm{refl}_a) :\equiv b$. The following are equivalent:

1. *the type $\sum_{x:A} B(x)$ is contractible,*

2. *for every $x : A$, the map $f(x) : a = x \to B(x)$ is an equivalence.*

In other words, a type family has contractible total space if and only if it is the family of identity types. This gives a sort of reciprocal to the statement of fact 4.7.5.

4.7.4 Higher n

The 1-types are types in which the path types $x = y$ are sets. Thinking of types as ∞-groupoids, the 1-types thus correspond to groupoids: their "hom-types" are discrete. This is the incarnation in homotopy type theory of fact 2.2.18 in classical homotopy theory. More generally, higher n-types can be thought as the homotopy type theoretic incarnation of n-groupoids.

4.7.5 Closure properties of n-types

When trying to show a type A is an n-type in practice, rather than using direct proofs it is usually easier to use the way the type A was constructed (for instance as a cartesian product, or a mapping type). Here are some results one can use to that end.

Fact 4.7.20 ([Uni13, Thm 7.1.9]). *Let A be a type, and $B : A \to \mathcal{U}$ a type family such that for every $x : A$, $B(x)$ is an n-type. Then $\prod_{x:A} B(x)$ is an n-type.*
In particular in the non-dependent case, if B is an n-type, so is $A \to B$.

Proof. First the case $n = -2$: suppose $B(x)$ is contractible for every $x : A$. Then mapping every $x : A$ to the center of contraction of $B(x)$ defines a map $f : \prod_{x:A} B(x)$. Given any other map $g : \prod_{x:A} B(x)$, then for any $x : A$ the contractibility of $B(x)$ yields a path $f(x) = g(x)$. By function extensionality, this gives $f = g$, which proves that $\prod_{x:A} B(x)$ is contractible.
The proof for higher n follows by induction on n using function extensionality. $\square$

Fact 4.7.21 ([Uni13, Cor 7.1.5]). *If $A \simeq B$ and A is an n-type, so is B.*

Proof. This follows directly from univalence and transport, but it is also true without univalence, see [Uni13, p. 7.1.5]. $\square$

Fact 4.7.22 ([Uni13, Thm 7.1.8]). *Let A be an n-type and $B : A \to \mathcal{U}$ a family of n-types. Then $\sum_{x:A} B(x)$ is an n-type.*

Proof. This follows by induction on n from the characterization of path types in dependent sums (fact 4.5.6) and fact 4.7.21. $\square$

As a corollary of facts 4.7.11, 4.7.20 and 4.7.22, we get:

Corollary 4.7.23. *Let A and B be propositions, then $A \simeq B$ is a proposition.*

Which, together with fact 4.7.15, yields

Corollary 4.7.24. *If A and B are propositions, then*

$$(A \leftrightarrow B) \simeq (A \simeq B)$$

Fact 4.7.25 ([Uni13, Thm 7.1.11]). *Let $n \geq -2$. The type n-Type is an $(n+1)$-type.*

Proof. The idea of the proof is as follows: given $(X, p), (Y, q) : n$-Type, by fact 4.7.13 we have

$$(X, p) = (Y, q) \quad \simeq \quad X = Y.$$

By univalence, $(X = Y) \simeq (X \simeq Y)$. The type $X \simeq Y$ *embeds* in the type $X \to Y$ (see section 4.9), and $X \to Y$ is an n-type since Y is itself an n-type and by fact 4.7.20. Finally, embeddings preserve the homotopy level (fact 4.9.10), so $X \simeq Y$ is itself an n-type, and so is $(X, p) = (Y, q)$ by fact 4.7.21, so n-Type is an $(n+1)$-type. $\square$

Corollary 4.7.26. hProp *is a set.*

Corollary 4.7.27. hSet *is a groupoid (i.e. a 1-type).*

4.8 Reasoning with equivalences

Throughout the rest of this chapter we will often construct equivalences by composing simpler equivalences together, often relying on a few useful and intuitive lemmas. We list a few of these lemmas here.

Lemma 4.8.1. *Let A be a type and $B, B' : A \to \mathcal{U}$ be families of types such that*

$$\prod_{x:A} (B(x) \simeq B'(x)).$$

Then

$$\sum_{x:A} B(x) \simeq \sum_{x:A} B'(x).$$

Lemma 4.8.2. *Let $f : A \simeq A'$ be an equivalence and $B : A' \to \mathcal{U}$ a family of types. Then*

$$\sum_{x:A} B(f(x)) \simeq \sum_{x:A'} B(x).$$

The next three lemmas witness that the formation of sum types is "associative and unital up to equivalence".

Lemma 4.8.3. *Let $B : 1 \to \mathcal{U}$ be a family of types, then*

$$\sum_{x:1} B(x) \simeq B(\star).$$

Lemma 4.8.4. *Let A be a type, then $A \times 1 \simeq A$.*

Lemma 4.8.5. *Let A be a type, $B : A \to \mathcal{U}$ and $C : \sum_{a:A} B(a) \to \mathcal{U}$. We have an equivalence*

$$\sum_{a:A} \sum_{b:B(a)} C((a,b)) \simeq \sum_{x : \sum_{a:A} B(a)} C(x)$$

An important corollary of lemma 4.8.3 and fact 4.7.3 is the following:

Corollary 4.8.6. *Let A be a contractible type, with center of contraction $a_0 : A$, and $B : A \to \mathcal{U}$ a family of types. Then*

$$\sum_{a:A} B(a) \simeq B(a_0).$$

4.9 The indexed/fibred equivalence in homotopy type theory

Recall from section 2.6 the correspondence in set theory between families of sets $(Y_x)_{x \in X}$ indexed by a set X and "fibred" maps $Y \to X$. The same kind of correspondence is true in homotopy type theory, and as we will see it is in some ways even better behaved than the set-theoretic one (remark 4.9.4).

Given an indexed type family $B : A \to \mathcal{U}$, the corresponding fibred type over A is the dependent sum type $\sum_{x:A} B(x)$, together with the projection $\pi_1 : \sum_{x:A} B(x) \to A$.

In the other direction, given a map $p : B \to A$ and an element $a : A$, we can take the fiber of p at a.

Definition 4.9.1. *Let $p : B \to A$ be a map between types, and $a : A$. The* fiber *of p at a, written* $\mathrm{fib}_p(a)$, *is the type*

$$\mathrm{fib}_p(a) :\equiv \sum_{x:B} f(x) = a$$

When the type A is a set, the type $(f(x) = a)$ is a proposition, so in a sense we can think of $\mathrm{fib}_p(a)$ as a "subtype" of B on those elements that satisfy this proposition. When A is not a set however, the possibly many elements in $f(x) = a$ transpire in the fiber $\mathrm{fib}_p(a)$. For instance, consider the map $f : 1 \to \mathcal{U}$ that maps the unique element $\star : 1$ to the type Bool of booleans. The fiber of f at Bool $: \mathcal{U}$ is the type $\sum_{x:1} f(x) = $ Bool, which is straightforwardly equivalent to the type Bool $=$ Bool, which by example 4.6.3 is itself equivalent to Bool. So the fiber of the map $f : 1 \to \mathcal{U}$ is a set with two elements, despite 1 having only one element.

Remark 4.9.2. In classical topology, the above definition of fiber corresponds more closely to the notion of *homotopy fiber*: a notion of fiber that is invariant under homotopy equivalence of topological spaces. Categorically speaking, the fiber of a morphism $f : X \to Y$ at a point $y \in Y$ is defined as the following pullback.

$$
\begin{array}{ccc}
f^{-1}(x) & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow f \\
1 & \underset{x}{\longrightarrow} & Y
\end{array}
$$

The homotopy fiber is defined using a *homotopy pullback* instead: a notion we will revisit in section 5.3 (see example 5.3.16).

The fiber operator defines a type family $\text{fib}_p : A \to \mathcal{U}$, so we get the following maps

$$\sum_{B:\mathcal{U}} (B \to A) \qquad\qquad (A \to \mathcal{U})$$

$$(B, p) \longmapsto (a \mapsto \text{fib}_p(a))$$

$$\sum_{x:A} F(x) \longleftarrow\! F$$

Fact 4.9.3 (*Indexed/fibred correspondence for types*, [Uni13, Thm 4.8.3]). *The above two maps are inverses of each other, so for every type A there is an equivalence*

$$\sum_{B:\mathcal{U}} (B \to A) \quad \simeq \quad (A \to \mathcal{U}).$$

Proof. First, let $p : B \to A$ be a map between types. We want to show that

$$\Big(\sum_{a:A} \text{fib}_p(a), \pi_1 \Big) =_{\sum_{X:\mathcal{U}} (X \to A)} (B, p).$$

By fact 4.5.6 and univalence, this reduces to constructing an equivalence $\theta : B \simeq \sum_{a:A} \text{fib}_p(x)$ such that for all $b : B$, $\pi_1(\theta(b)) = p(b)$. Using reasoning by equivalence in sum types (section 4.8), we have the following:

$$\begin{aligned}
\sum_{a:A} \text{fib}_p(a) :&\equiv \sum_{a:A} \sum_{b:B} p(b) = a \\
&\simeq \sum_{b:B} \Big(\sum_{a:A} p(b) = a \Big) && \text{(reordering terms)} \\
&\simeq \sum_{b:B} 1 && \text{(by fact 4.7.5 and fact 4.7.3)} \\
&\simeq B && \text{(by lemma 4.8.4)}
\end{aligned}$$

Taking the inverse of this chain of equivalences, we get an equivalence $\theta : B \simeq \sum_{x:A} \text{fib}_p(x)$ which computes to

$$\theta(b) :\equiv (b, p(b), \text{refl}_{p(b)})$$

so that $\pi_1(\theta(b)) :\equiv b$, which gives our desired result.

Second, let $F : A \to \mathcal{U}$ be a family of types over A. We need to show that $F = \text{fib}_{\pi_1}$, where $\pi_1 : \sum_{a:A} F(a) \to A$. By function extensionality (fact 4.6.4) and univalence, this reduces to showing that for every $a : A$, we can find an equivalence

$$F(a) \simeq \sum_{x : \sum_{a:A} F(a)} \pi_1(x) = a.$$

As before we reason by chain of equivalences:

$$\begin{aligned}
\sum_{x : \sum_{a:A} F(a)} \pi_1(x) = a &\simeq \sum_{u:A} \sum_{v:F(a)} \pi_1((u,v)) = a && \text{(lemma 4.8.5)} \\
&\equiv \sum_{u:A} \sum_{v:F(u)} u = a && \text{(def. of } \pi_1) \\
&\simeq \sum_{\chi : \sum_{u:A} u = a} F(\pi_1(\chi)) = a && \text{(reordering terms)} \\
&\simeq F(\pi_1((a, \text{refl}_a))) && \text{(by fact 4.7.5 and corollary 4.8.6)} \\
&\equiv F(a)
\end{aligned}$$

which gives us the desired result. □

Remark 4.9.4. In set theory, the equivalence between sets over X and families of sets indexed by X is not bijective, as it is only an equivalence of sets up to isomorphisms. In homotopy type theory however, since "isomorphism" is replaced by equivalence, and thanks to univalence, this correspondence for types can be stated as an equivalence itself without even introducing category theory.

Fibers and n-types. Let $f : B \to A$ be a map of types. If f has contractible fibers, fact 4.9.3 implies that

$$B \simeq \sum_{x:A} 1 \simeq A$$

so that f is an equivalence. Having contractible fibers is actually another possible definition for being an equivalence.

Fact 4.9.5 ([Uni13, Sec 4.5]). *Let $f : X \to Y$ be a map of types. There is an equivalence of types*

$$\text{isEquiv}\,(f) \simeq \prod_{y:Y} \text{isContr}\,(\text{fib}_f(y)).$$

More generally, fact 4.9.3 implies that type families over A with values in n-types correspond to maps $B \to A$ whose fibers are n-types.

Definition 4.9.6 ([Uni13, Def 7.6.1]). A map $f : A \to B$ is said to be *n-truncated* if its fibers are n-types.

The following fact allows an inductive characterization of n-truncated maps similar to the inductive definition for n-types.

Fact 4.9.7. *A map $f : A \to B$ is $(n+1)$-truncated if and only if for every $x, y : A$, the action on paths of f*

$$\text{ap}_f : x =_A y \to f(x) =_B f(y)$$

is n-truncated.

In the case $n = -1$, the (-1)-truncated maps are also called *embeddings*.

Definition 4.9.8. A map $f : A \to B$ is said to be an *embedding* if for every $x, y : A$, the action on paths of f

$$\text{ap}_f : x =_A y \to f(x) =_B f(y)$$

is an equivalence. In that case we write $f : A \hookrightarrow B$.

Remark 4.9.9. Thinking of groupoids as types, embeddings correspond to fully faithful functors.

Embeddings are the right generalization of injective maps from set theory to homotopy type theory. Indeed a map $f : A \to B$ between sets (in the sense of definition 4.7.16) is an embedding if and only if for every $x, y : A$, there is a map $f(x) = f(y) \to x = y$, as the latter induces a logical equivalence between $f(x) = f(y)$ and $x = y$ and thus an equivalence of types by virtue of fact 4.7.15 and A and B being sets. Categorically, embeddings can be thought of as the *monomorphisms* or *subobjects* in the "higher category of types".

Applying the correspondence of fact 4.9.3 to embeddings, we see that for a fixed B, embeddings $A \hookrightarrow B$ correspond exactly to maps $B \to \text{hProp}$. In that sense, hProp acts as the *subobject classifier* in homotopy type theory. The fact that isProp (X) is a proposition also entails that the forgetful

map $\pi_1 : \text{hProp} \to \mathcal{U}$ is itself and embedding, so that equality between propositions is the same as equality between the underlying types. More generally we have embeddings

$$\text{hProp} \hookrightarrow \mathcal{U}$$
$$\text{hSet} \hookrightarrow \mathcal{U}$$
$$n\text{-Type} \hookrightarrow \mathcal{U}$$

A nice property of embeddings is that they preserve the homotopy level of types.

Fact 4.9.10 ([Uni13, Thm 7.1.6])**.** *Let $f : A \hookrightarrow B$ be an embedding such that B is an n-type for $n \geq 1$. Then A is an n-type.*

Note that the hypothesis $n \geq 1$ is important as there is an embedding $\varnothing \hookrightarrow 1$ and the empty type is clearly not contractible. Combining fact 4.9.10 and fact 4.7.11, we get the following.

Fact 4.9.11. *Let A be a type and B be an n-type for $n \geq 1$. Then the type of equivalences $A \simeq B$ is itself an n-type.*

Corollary 4.9.12 ([Uni13, Thm 7.1.11])**.** *The type n-Type is an $(n+1)$-type.*

In particular, the type hProp of propositions is a set and the type hSet of sets is a 1-type (i.e. a groupoid).

Remark 4.9.13. One might expect that there be only two propositions, i.e. that the canonical map

$$\text{Bool} \to \text{hProp}$$
$$\text{false} \mapsto \varnothing$$
$$\text{true} \mapsto 1$$

be an equivalence. However this statement is *independent* from homotopy type theory. Indeed, homotopy type theory is based on type theory, a foundation for *constructive mathematics*, and the above map being an equivalence is akin to a statement of the *law of excluded middle*, which is not provable constructively. We will not go into more detail about this here, but the interested reader can learn more about this in [Uni13, Sec 3.4].

Remark 4.9.14. Just like hProp acts as a subobject classifier in homotopy type theory, hSet acts as a kind of classifier for families of sets. Such a *set classifier* cannot exist in the category of sets as it would need to be a groupoid. As such set classifiers cannot generally exist in 1-category theory, but they may exist in bicategories, especially 2-toposes.

More generally, the type $\mathcal{U}$ may be thought as a general *object classifier*, as fact 4.9.3 precisely states that every object over A is classified by a map $A \to \mathcal{U}$ (see [Uni13, Sec 4.8] for more on this).

4.10 Higher inductive types

In type theory, we saw how to define types axiomatically via induction principles (section 4.3). From what we saw, every type we could define in this way ended up being sets, and we had to use univalence to find an example of a non-set type (namely, the universe). In this section we explore another axiomatic approaches to building higher-dimensional types: higher inductive types. The general theory of higher inductive types is described in chapter 6 of [Uni13] with various examples, here we will only present one prototypical example with the goal of making the reader familiar with the main ideas of the theory.

Like usual inductive types, higher inductive types are specified by a list of constructors, from which one deduces an induction principle. However while the constructors of an inductive type A specify how to construct elements in A, constructors in a higher inductive type may also construct paths between elements.

Definition 4.10.1. The *circle* S^1 is a higher inductive type with constructors

$$\text{base} : S^1$$
$$\text{loop} : \text{base} =_{S^1} \text{base}$$

In other words, the circle is freely generated by a point base $: S^1$ together with a path loop $: \text{base} = \text{base}$.
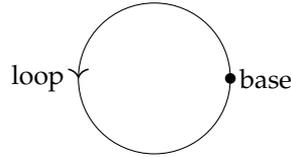
The induction principle for the circle states that given a type family $B : S^1 \to \mathcal{U}$, a point $b : B(\text{base})$ and a dependent path $p : b =_B^{\text{loop}} b$, there is a dependent map

$$\text{ind}_{S^1}(B, b, p) : \prod_{x:S^1} B(x)$$

with computation rules

$$\text{ind}_{S^1}(B, b, p)(\text{base}) \equiv b$$
$$\text{apd}_{\text{ind}_{S^1}(B,b,p)}(\text{loop}) = p$$

The second computation rule uses typal equality rather than judgmental equality for technical reasons that we will not get into here, and we refer the interested reader to chapter 6 of [Uni13].

In the case where B is a constant type family, by fact 4.4.19 we can just work with non-dependent paths, ask for $p : b = b$ instead of $p : b =_B^{\text{loop}} b$ and get a computation rule

$$\text{ap}_{\text{ind}_{S^1}(B,b,p)}(\text{loop}) = p.$$

In the path type base $=_{S^1}$ base, we can find elements $\text{refl}_{\text{base}}$, loop, loop $\cdot$ loop, loop^{-1}, ... Since S^1 is "freely generated" by loop, it is to be expected that all these paths are distinct, and indeed they are. First we can easily prove that $\text{refl}_{\text{base}} \neq$ loop using univalence ([Uni13, Lem 6.4.1]).

Proof. Consider the map $B : S^1 \to \mathcal{U}$ that maps base to the type Bool and loop to the path corresponding by univalence to the swapping self-equivalence ua (swap) : Bool = Bool described in example 4.6.3. By definition of ap_B, we have $\text{ap}_B(\text{refl}_{\text{base}}) = \text{refl}_{\text{Bool}}$, and by computation rule for the induction principle of the circle we have that $\text{ap}_B(\text{loop}) = \text{ua}(\text{swap})$. As we have shown in example 4.6.3 that $\text{refl} \neq \text{ua}(\text{swap})$, we deduce $\text{refl}_{\text{base}} \neq$ loop. $\square$
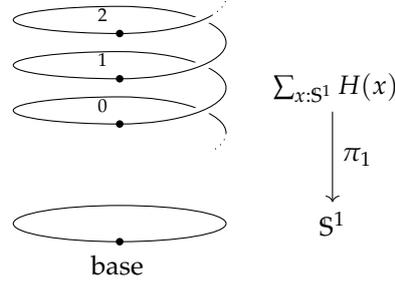
Figure 4.1: Spatial representation of the projection from the helix to the circle (from [Uni13, Fig 8.1]).

We can actually prove the way stronger result that

$$(\text{base} =_{\mathbb{S}^1} \text{base}) \simeq \mathbb{Z}$$

where the type $\mathbb{Z}$ of integers can be defined as an inductive type similarly to $\mathbb{N}$. Multiple proofs of this result are detailed in section 8.1 of [Uni13], here we repeat a nice, visual proof.

Write $\text{suc} : \mathbb{Z} \simeq \mathbb{Z}$ for the $n \mapsto n + 1$ self-equivalence of $\mathbb{Z}$. Consider the type family $H : \mathbb{S}^1 \to \mathcal{U}$ defined by

$$H(\text{base}) :\equiv \mathbb{Z}$$
$$\text{ap}_H(\text{loop}) := \text{ua}(\text{suc})$$

and consider its total space $\sum_{x:\mathbb{S}^1} H(x)$. For a general $x : \mathbb{S}^1$, $H(x)$ is a kind of "twisted" version of the type $\mathbb{Z}$. A path in $\sum_{x:\mathbb{S}^1} H(x)$ from (base, m) to (base, n) corresponds (by fact 4.5.6) to a pair of a path $p : \text{base} = \text{base}$ and an equality $\text{transport}_p^H(m) = n$. Taking $p \equiv \text{loop}$, since ua is defined as an inverse to transport, we get that $\text{transport}_{\text{loop}}^H(m) = m + 1$, so we would need an equality $m + 1 = n$.

The picture to have in mind is the following: the type $\sum_{x:\mathbb{S}^1} H(x)$ projects onto $\mathbb{S}^1$, and the fiber over any point $x : \mathbb{S}^1$ looks like a copy of $\mathbb{Z}$. Those fibers are pasted together along the circle in way that, when making a complete turn in $\sum_{x:\mathbb{S}^1} H(x)$ over the path loop in $\mathbb{S}^1$, we land back one number higher than we started at: H describes a *helix* above the circle, see fig. 4.1. As can be seen in the picture, the total space $\sum_{x:\mathbb{S}^1} H(x)$ is a helix, hence a twisted line, and as such it should be contractible. We won't go into the details of the proof here, as always the interested reader may check [Uni13, Sec 8.1.6], but we simply state the result here and its consequence.

Fact 4.10.2 ([Uni13, Thm 8.1.15]). *The type $\sum_{x:\mathbb{S}^1} H(x)$ is contractible.*

Applying the fundamental theorem of identity types (fact 4.7.19), we get the following corollary.

Corollary 4.10.3 ([Uni13, Cor 8.1.16]). *For every $x : \mathbb{S}^1$, there is an equivalence*

$$H(x) \simeq (\text{base} = x).$$

In particular, $(\text{base} = \text{base}) \simeq \mathbb{Z}$.

Higher dimensional types. When defining higher inductive types, we can define constructors with values in arbitrarily high-dimensional path types.

Example 4.10.4. The *sphere* S^2 has constructors

$$\mathrm{base} : S^2$$
$$\mathrm{surf} : \mathrm{refl}_{\mathrm{base}} = \mathrm{refl}_{\mathrm{base}}$$

In this way, it is possible to define higher dimensional spheres as types in homotopy type theory, and more generally one can represent arbitrary CW-complexes as types. In particular, every groupoid (in the usual sense of set theory) can be represented as a type, more specifically a 1-type.

4.11 Truncation

Given a topological space X, one can consider the set of its path-components: this set $\pi_0(X)$ can be thought as the 0-th homotopy set (rather than homotopy group) of X, as its elements are the 0-dimensional paths in X (i.e. the points of X) up to homotopy (i.e. up to being connected by a path). It is a homotopical invariant: $\pi_0(X)$ depends only on the homotopy type of X (see section 2.2).

Thinking of types as representing homotopy types, we would like to be able to do the same kind of construction. The set of path-components has a universal property that is easy to translate in homotopy type theory: given a type X, the type $\pi_0(X)$ should be a set, together with a map $X \to \pi_0(X)$ that is *universal* among maps from X to sets. More generally, we define a type $\|X\|_n$ as the universal n-type equipped with a map from X.

Axiom 4.11.1. For every type X and $n \geq -2$, there is an n-type $\|X\|_n$ called the *n-truncation* of X, equipped with a map

$$| - |_n : X \to \|X\|_n$$

such that for every n-type Y, the post-composition map

$$(\|X\|_n \to Y) \to (X \to Y)$$
$$f \mapsto f \circ | - |_n$$

is an equivalence.

Remark 4.11.2. By fact 4.9.5, we can rephrase the above condition as: for every $f : X \to Y$, the type $\sum_{g:\|X\|_n \to Y} f = g \circ | - |_n$ is contractible. Since contractibility is the right notion of uniqueness in homotopy type theory, this should be thought as

Every map f from X to an n-type factors uniquely through $\|X\|_n$.

In [Uni13], general n-truncations are built as higher inductive types. Here we introduce them axiomatically instead to not get lost in technical details of the general definition, and focus on low-dimensional examples, although we mention an important consequence of the inductive definition: the elimination principle for truncations.

Theorem 4.11.3 (*Elimination principle for truncations*, [Uni13, Thm 7.3.2]). *Let X be a type, $n \geq -2$, and $Y : \|X\|_n \to \mathcal{U}$ a type family over $\|X\|_n$ such that for every $a : \|X\|_n$, $Y(a)$ is an n-type, and*

$$g : \prod_{x:X} Y(|x|_n)$$

a dependent map. Then there is a dependent map

$$f : \prod_{a:\|X\|_n} Y(a)$$

with computation rule

$$f(|x|_n) :\equiv g(x)$$

for every $x : X$.

4.11.1 Propositional truncation

Definition 4.11.4. In the case $n = -1$, the type $\|X\|_n$ is called the *propositional truncation* of X.

The propositional truncation of a type X can be define as a higher inductive type with the following constructors:

$$| - |_{-1} : X \to \|X\|_{-1}$$
$$\text{squash}_{-1} : \prod_{x:\|X\|_{-1}} \prod_{y:\|X\|_{-1}} x = y$$

The constructor squash_{-1} is *recursive*: just like the constructor $\text{suc} : \mathbb{N} \to \mathbb{N}$, it refers as inputs to elements of the type it is constructing. This constructor "forces" every element in the type $\|X\|_{-1}$ to be equal, it "squashes" them all together. As one can easily see, we have $\text{squash}_{-1} : \text{isProp}(\|X\|_{-1})$ by definition, and thus it is not hard to prove that this definition of $\| - \|_{-1}$ satisfies the axiomatic definition of (-1)-truncation described above.

Propositional truncation allows us to make statements that "forget witnesses" even though we are working in constructive foundations. Giving an element $x : \|X\|_{-1}$ is a way to state that X is *inhabited* without specifying an actual element of X.

Taking the example of the helix family $H : S^1 \to \mathcal{U}$ from section 4.10, it is possible to show that

$$\neg \prod_{x:S^1} H(x)$$

Proof. Suppose given $f : \prod_{x:S^1} H(x)$. Write $m :\equiv f(\text{base}) : \mathbb{Z}$. We have $\text{apd}_f(\text{loop}) : \text{loop}_*(m) = m$, i.e. $m + 1 = m$, which leads to a contradiction. $\qquad\square$

More intuitively, such a map would correspond to a global section of the helix, and such a section cannot be continuous. However, we can show the following:

$$\prod_{x:S^1} \|H(x)\|_{-1}$$

This statement should be read not as "there is a dependent map from $x : S^1$ to $H(x)$", but as "for every $x : S^1$, *there exists* some element in $H(x)$". The authors of [Uni13] use the terminology of X being *merely inhabited*.

4.11.2 Connectedness

Recall from the discussion following definition 4.7.1 how the definition of contractibility of X looks like it is simply stating that any pair of points in X can be joined by a path — i.e. that X is connected — but that it also entails that any pair of paths can be joined by a higher path and so on. Well, using the propositional truncation, we can actually define connectedness:

Definition 4.11.5. A type X is *connected* if:

1. it is merely inhabited, i.e. there is some $x : \|X\|_{-1}$,

2. for every $x, y : X$, there is some $p : \|x = y\|_{-1}$.

Example 4.11.6. The circle S^1 is connected:

1. We have $|\operatorname{base}|_{-1} : \|S^1\|_{-1}$.

2. We want to prove that $\prod_{x:S^1} \prod_{y:S^1} \|x = y\|_{-1}$. For that we can use the induction principle for the circle on x and then y, and in both cases the action on loop can be trivially defined as $\prod_{y:S^1} \|x = y\|_{-1}$ and $\|x = y\|_{-1}$ are propositions. So we only need to check that $\|\operatorname{base} = \operatorname{base}\|_{-1}$, which is trivially satisfied by $|\operatorname{refl}_{\operatorname{base}}|_{-1}$.

4.11.3 Set truncation

Definition 4.11.7. In the case $n = 0$, the type $\|X\|_n$ is called the *set truncation* of X.

Given a type X, $\|X\|_0$ can be thought of as the set of "connected path-components" of X, the function $|-|_0 : X \to \|X\|_0$ mapping every point $x : X$ to its connected path-component.

Example 4.11.8. The circle has only one path-component, so that

$$\|S^1\|_0 \simeq 1$$

Actually, we can also use set truncation to define connectedness:

Fact 4.11.9. *A type X is connected if and only if its set truncation $\|X\|_0$ is contractible.*

This is a consequence of the more general behaviour of path types with respect to truncation:

Fact 4.11.10 ([Uni13, Thm 7.3.12])**.** *Let X be a type, $x, y : X$ and $n \geq -2$. There is an equivalence*

$$\|x =_X y\|_n \simeq \left(|x|_{n+1} =_{\|X\|_{n+1}} |y|_{n+1} \right)$$

4.12 Homotopy groups

We mentioned in section 2.2 the theory of homotopy groups as one of the fundamental objects of study in homotopy theory. We will not use the general theory of homotopy groups in homotopy type theory in this thesis, but due to their significance we still briefly mention their definition.

The closest thing to homotopy groups that we have defined thus far in homotopy type theory are the iterated loop spaces $\Omega^n(X, x)$ of a pointed type (X, x). The elements of the type $\Omega^n(X, x)$ correspond to n-dimensional paths in X, but the type itself is not a set, it still contains homotopical data above dimension n. To get actual homotopy groups, we need to use set truncation:

Definition 4.12.1 ([Uni13, Def 8.0.1])**.** Let (X, x) be a pointed type. The n-th *homotopy group* of (X, x) is the set

$$\pi_n(X, x) :\equiv \|\Omega^n(X, x)\|_0.$$

By convention $\pi_0(X, x) :\equiv \|X\|_0$ does not depend on $x : X$, and when $n \geq 1$ the set $\pi_n(X, x)$ inherits a group structure from path concatenation in $\Omega^n(X, x)$.

Example 4.12.2. By corollary 4.10.3, we have $\Omega(S^1, \text{base}) \simeq \mathbb{Z}$. Together with combining example 4.11.8 and fact 4.11.9 for the case $n = 0$, we get

$$\pi_1(S^1, \text{base}) \simeq \mathbb{Z}$$
$$\pi_n(S^1, \text{base}) \simeq 1 \text{ when } n \geq 2 \text{ or } n = 0$$

Remark 4.12.3. Homotopy groups can be very hard to compute in general. For instance, the sphere S^2 has infinitely non-trivial homotopy groups, even though it is generated by one point and a single 2-dimensional path. The study of the homotopy groups of higher-dimensional spheres is an ongoing research project in homotopy theory since the very birth of the subject.

Every map $f : X \to Y$ induces for every $x : X$ and $n : \mathbb{N}$ a map $\pi_n(X, x) \to \pi_n(Y, f(x))$, which is a group morphism whenever $n \geq 1$. In classical homotopy theory, Whitehead's theorem (fact 2.2.15) states that if every one of these induced maps is an isomorphism, then f must be a homotopy equivalence (when X and Y are CW-complexes). One may wonder if this result can be proven in homotopy type theory, and turns that that this is *not* the case. This is because homotopy type theory has models in arbitrary ∞-toposes ([Shu19]), just like constructive set theory has models in arbitrary toposes, and Whitehead's theorem does not hold in every such ∞-topos. Informally, this is because in some ∞-toposes there exist objects that are "∞-connected" but no contractible. The theorem is true however when X and Y are n-types for some $n \geq -2$. For more on this, see [Uni13, Sec 8.8].

Chapter 5

Linear logic in Homotopy Type Theory

To be able to define higher categorical models of linear logic in the setting of homotopy type theory, we need a working theory of higher categories in HoTT. Relying on the intuition that ∞-category theory is a version of category theory defined *internally to* homotopy theory rather than set theory, we see that higher categories in HoTT should have an underlying *type* of objects, and for every pair of objects a *type* of morphisms, together with identities and a composition operation that is associative and unital. So far this can all be done by stating the usual definition of categories and replacing sets by types where appropriate.

However an issue rises when we look at the associativity and unitality conditions: in set theory they are propositional, they correspond to equalities between elements in a set. In homotopy type theory, associativity and unitality is instead the *data* of a *choice of identities*: for instance associativity is witnessed by an element in the type

$$((h \circ g) \circ f) =_{\mathrm{Hom}_{\mathcal{C}}(x,y)} (h \circ (g \circ f)).$$

As already well understood for instance by looking at the set-theoretic definition of bicategories (section 2.1), this data must itself be subject to further axioms called *coherence conditions*, and these coherence conditions themselves will be additional data, itself subject to further coherence conditions, and so on. Since in general an arbitrary type may not be an *n-type* for any $n \geq -2$, there is no reason for these conditions to become propositional at any point.

Perhaps surprisingly, the problem of finding a type-theoretic definition that encapsulates all these higher coherences is currently open, and believed by many experts in the field to be unsolvable without extending homotopy type theory itself [Buc19]. Because of this, throughout this chapter we will work with a weaker notion of higher categories, called *wild categories*. Wild categories were introduced by Paolo Capriotti and Nicolai Kraus in [CK18], and as already hinted at in the introduction to chapter 4, they are what we get when we simply discard higher coherences while still work in a homotopical setting.

A formalization of some of the results of this chapter in the proof assistant Cubical Agda is available at `https://github.com/elies-h/ll-polynomials`.

Remark 5.0.1. Even though it is unknown how to define ∞-categories in plain HoTT, it is possible to do so in some experimental extensions of the type theory such as the one of [AFS21; All23], in which it is even possible to construct the ∞-category of spans of types.

We begin by developing what we will need of the theory of wild categories: the basic definitions of wild categories, functors and natural transformations in section 5.1, the theory of symmetric monoidal structures and (co)monads in section 5.2, some theory of limits and colimits in section 5.3, and how to truncate wild categories to recover ordinary categories in section 5.4. Leveraging all this theory, we can finally define wild categorical models of linear logic in section 5.5.

The next sections are dedicated to building a (family of) model(s) of wild categorical models of linear logic based on spans of types: we first define wild categories of spans in section 5.6, where we also show that the $\mathrm{Span}\,(-)$ construction is functorial, similarly to the functoriality of the $\mathrm{Rel}\,(-)$ construction in section 3.6. Using this functoriality, we lift a monad on the universe of types $\mathcal{U}$ to a comonad on $\mathrm{Span}\,(\mathcal{U})$ in section 5.7, showing that we get this way a wild categorical model of linear logic (theorem 5.7.33). In section 5.8, we show that the wild Kleisli category for this comonad is precisely the wild category of polynomial functors from [Fin+21] (theorem 5.8.17), so that our model is indeed an extension of theirs. Finally, we investigate in section 5.9 under which conditions does this model extend to a wild bicategorical model of linear logic, the main result being theorem 5.9.10.

5.1 Wild categories, functors, and natural transformations

In homotopy type theory, (wild) categories should have an underlying type of objects, rather than an underlying set of objects. Because of this, there should be two canonical notions of "isomorphism" between objects x and y in a (wild) category $\mathcal{C}$:

1. the one given by the categorical structure, i.e. the morphisms $f : x \to y$ in $\mathcal{C}$ that are invertible,

2. the one given by the type-theoretic structure, i.e. the paths $p : x = y$ in the type of objects.

In general, these two notions have no reason to coincide, and we speak of (wild) *pre*categories — reserving the name of (wild) categories only when they do coincide, by analogy with the difference between preorders and orders (see remark 5.1.9).

Definition 5.1.1 ([CK18]). A *wild precategory* $\mathcal{C}$ is the data of

- a type $\mathrm{Ob}\,\mathcal{C} : \mathcal{U}$ of *objects*,

- a family of types $\mathrm{Hom}_{\mathcal{C}} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C} \to \mathcal{U}$ of *morphisms*,

- for every $x : \mathrm{Ob}\,\mathcal{C}$, a distinguished morphism $\mathrm{id}_x : \mathrm{Hom}_{\mathcal{C}}(x, x)$,

- for all $x, y, z : \mathrm{Ob}\,\mathcal{C}$, a composition operation

$$- \circ - : \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

- for all $x, y : \mathrm{Ob}\,\mathcal{C}$ and $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$, equalities

$$\lambda_f : \mathrm{id}_y \circ f = f$$
$$\rho_f : f \circ \mathrm{id}_x = f$$

called the *left unitor and right unitor*

- for all $x, y, z, t : \mathrm{Ob}\,\mathcal{C}$, $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$, $g : \mathrm{Hom}_{\mathcal{C}}(y, z)$ and $h : \mathrm{Hom}_{\mathcal{C}}(z, t)$, an equality

$$\alpha_{h,g,f} : (h \circ g) \circ f = h \circ (g \circ f)$$

called the *associator*

We will often abuse notation and write $x : \mathcal{C}$ directly instead of $x : \mathrm{Ob}\,\mathcal{C}$. As in usual category theory, we often write $f : x \to y$ or $f : \mathrm{Hom}\,(x, y)$ instead of $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$, omitting the precategory $\mathcal{C}$ when it is clear from context.

Definition 5.1.2. A wild precategory is a *precategory* if its type of objects is a 1-type and its types of morphisms are 0-types.

In a precategory, since the Hom -types are 0-types, the data of the equalities λ, ρ and α is actually propositional, so that unitality and associativity is just a proposition and not a structure. In general wild precategories, the equalities λ, ρ and α are additional structure on the composition law, and not just propositional axioms. In particular, there could be multiple non-equivalent wild precategories with the same objects, morphisms, identities and composition law, but different unitality and associativity witnesses.

Example 5.1.3. Given a universe of types $\mathcal{U}$, there is a wild precategory also written $\mathcal{U}$, whose type of objects is $\mathcal{U}$ and $\mathrm{Hom}_{\mathcal{U}}(x, y)$ is the ordinary function type $x \to y$. Composition is the ordinary composition of functions, identities are the identity functions, and unitality and associativity are given by refl.

Definition 5.1.4. Let $f : x \to y$ be a morphism in a wild precategory $\mathcal{C}$. We say that f is an *isomorphism* when it admits both a left and a right inverse [CK18]. We write

$$\mathrm{isIso}\,(f) := \Big(\sum_{g:y\to x} g \circ f = \mathrm{id}_x \Big) \times \Big(\sum_{g:y\to x} f \circ g = \mathrm{id}_y \Big)$$

for the corresponding type. We write $\mathrm{Iso}_{\mathcal{C}}\,(x, y)$, $x \simeq_{\mathcal{C}} y$ or simply $x \simeq y$ for the type $\sum_{f:x\to y} \mathrm{isIso}\,(f)$ of isomorphisms from x to y.

Proposition 5.1.5. *In the setting of definition 5.1.4, the type* $\mathrm{isIso}\,(f)$ *is a proposition.*

Proof. First suppose given $((g, p), (h, q)) : \mathrm{isIso}\,(f)$. Then we have

$$g = g \circ (f \circ h) = (g \circ f) \circ h = h.$$

Hence g is both a left and right inverse to f. Thus the type $\Big(\sum_{g:y\to x} g \circ f = \mathrm{id}_x \Big)$ is equivalent to $\sum_{g':(y\to x)} g' = g$, which is contractible, and similarly for the second component of $\mathrm{isIso}\,(f)$. We proved that $\mathrm{isIso}\,(f) \to \mathrm{isContr}\,(\mathrm{isIso}\,(f))$, which implies that $\mathrm{isIso}\,(f)$ is a proposition. $\qquad\square$

Note that a similar property is shown in [Uni13, Lemma 9.1.3] for precategories, but the proof relies on the fact that hom types in a precategory are sets. In fact, we define isIso by asking independently for a left and a right inverse (and not, say, for a single morphism that's at the same time a left and a right inverse) precisely so that $\mathrm{isIso}\,(f)$ is a proposition. This issue is closely related to that of defining a "good" notion of equivalence between types [Uni13, Chapter 4], and indeed applying definition 5.1.4 to example 5.1.3 yields the notion of "bi-invertible morphism" defined in [Uni13].

In a wild precategory $\mathcal{C}$, when the two competing notions of "isomorphism" between objects given by isomorphisms in $\mathcal{C}$ and the equalities in the type $\mathrm{Ob}\,\mathcal{C}$ coincide, we say that $\mathcal{C}$ is univalent.

Definition 5.1.6. Let $\mathcal{C}$ be a wild precategory. For every object $x : \mathcal{C}$, the identity morphism $\mathrm{id}_x : x \to x$ is an isomorphism (via λ or ρ), so by path induction we get for all $x, y : \mathcal{C}$ a map $\mathrm{idtoiso} : x = y \to x \simeq y$.

We say that $\mathcal{C}$ is *univalent* if this canonical map $\mathrm{idtoiso} : x = y \to x \simeq y$ is an equivalence for all $x, y : \mathcal{C}$, and write $\mathrm{ua} : x \simeq y \to x = y$ for an inverse map. A univalent wild precategory is called a *wild category*. Similarly, a univalent precategory is called a category.

Remark 5.1.7. Since being an equivalence is a proposition (fact 4.7.15) and propositions are closed under dependent product types (fact 4.7.20), being univalent is also a proposition.

Example 5.1.8. Assuming function extensionality, the univalence axiom precisely states that the wild precategory $\mathcal{U}$ of example 5.1.3 is univalent (up to the fact that the traditional definition of equivalence of types is equivalent to that of biinvertible equivalence, see [Uni13, Section 4.5]).

Remark 5.1.9. A wild precategory $\mathcal{C}$ whose hom-types are propositional can be thought of as a preorder on a type. In that case, being univalent is akin to the preorder being antisymmetric, i.e. a partial order. More precisely, in that context being univalent is equivalent to $\mathrm{Ob}\,\mathcal{C}$ being a set and $\mathrm{Hom}\,[\mathcal{C}]$ being an antisymmetric relation. In particular, if $\mathrm{Ob}\,\mathcal{C}$ is already known to be a set, then univalence is equivalent to antisymmetry.

This is the reason for the prefix "pre" in "precategory", just as a (partial) order is a preorder that is antisymmetric, a category is a precategory that is univalent.

Definition 5.1.10. Let $\mathcal{C}$ be a wild precategory. Its *opposite category*, written $\mathcal{C}^{\mathrm{op}}$, is the wild precategory with the same objects as $\mathcal{C}$, morphisms given by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) :\equiv \mathrm{Hom}_{\mathcal{C}}(y, x),$$

and identities, composition, unitors and associators inherited from $\mathcal{C}$.

Proposition 5.1.11. *A wild precategory $\mathcal{C}$ is univalent if and only if its opposite $\mathcal{C}^{\mathrm{op}}$ is univalent.*

Proof. Unfolding the definition of $\mathrm{Iso}_{\mathcal{C}}(x, y)$, we have a judgmental equality

$$\mathrm{Iso}_{\mathcal{C}}(x, y) \equiv \mathrm{Iso}_{\mathcal{C}^{\mathrm{op}}}(y, x).$$

Moreover, we can explicitly define an equivalence of types

$$\mathrm{Iso}_{\mathcal{C}}(x, y) \simeq \mathrm{Iso}_{\mathcal{C}}(y, x)$$
$$(f, ((g, p), (h, q))) \mapsto (g, ((f, r), (f, p)))$$

where r is obtained by transporting q through an equality $g = h$ (see the proof of proposition 5.1.5). It follows from the propositionality of isIso (proposition 5.1.5) that the same map in the other direction defines an inverse to this map, thus proving it is an equivalence.

In summary we get an equivalence

$$\mathrm{Iso}_{\mathcal{C}}(x, y) \simeq \mathrm{Iso}_{\mathcal{C}^{\mathrm{op}}}(x, y),$$

so the canonical map $x = y \to \mathrm{Iso}_{\mathcal{C}}(x, y)$ is an equivalence if and only if the canonical map $x = y \to \mathrm{Iso}_{\mathcal{C}^{\mathrm{op}}}(x, y)$ is an equivalence. $\square$

Definition 5.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be wild precategories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of:

- a map $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$

- for every $x, y : \mathrm{Ob}\,\mathcal{C}$, a map $F : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$,

- and equalities:

$$\iota_{F,x} : F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$$
$$\xi_{F,g,f} : F(g \circ f) = F(g) \circ F(f)$$

witnessing the compatibility with identities and composition of the action of F on morphisms,

We write $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ for the type of functors from $\mathcal{C}$ to $\mathcal{D}$.

Remark 5.1.13. Although we define wild precategories and functors using lists of data for readability, we could equally describe them using the usual notation for dependent types. For instance for functors, we get the following cumbersome description:

$$\mathrm{Fun}\,(\mathcal{C}, \mathcal{D}) :\equiv \sum_{F:\mathrm{Ob}\,\mathcal{C}\to\mathrm{Ob}\,\mathcal{D}} \sum_{\overline{F}:\,\prod_{x,y:\mathrm{Ob}\,\mathcal{C}}\,\mathrm{Hom}_{\mathcal{C}}(x,y)\to\mathrm{Hom}_{\mathcal{D}}(F(x),F(y))}$$

$$\Big(\prod_{x:\mathrm{Ob}\,\mathcal{C}} \overline{F}(x, x, \mathrm{id}_x) = \mathrm{id}_{F(x)} \Big) \times$$

$$\Big(\prod_{x,y,z:\mathrm{Ob}\,\mathcal{C}} \prod_{f:\mathrm{Hom}_{\mathcal{C}}(x,y)} \prod_{g:\mathrm{Hom}_{\mathcal{C}}(y,z)} \overline{F}(x, z, g \circ f) = \overline{F}(y, z, g) \circ \overline{F}(x, y, f) \Big)$$

Definition 5.1.14. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be wild precategories, and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ be functors. Their composition $G \circ F : \mathcal{C} \to \mathcal{E}$ is defined as:

- for $x : \mathrm{Ob}\,\mathcal{C}$, $(G \circ F)(x) :\equiv G(F(x))$,

- for $x, y : \mathrm{Ob}\,\mathcal{C}$ and $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$, $(G \circ F)(f) :\equiv G(F(f))$,

- for $x : \mathrm{Ob}\,\mathcal{C}$, $\iota_{G\circ F,x} :\equiv \mathrm{ap}_G(\iota_{F,x}) \cdot \iota_{G,F(x)}$, i.e. the concatenation of

$$G(F(\mathrm{id}_x)) \overset{\mathrm{ap}_G(\iota_{F,x})}{=\!=\!=\!=} G(\mathrm{id}_{F(x)}) \overset{\iota_{G,F(x)}}{=\!=\!=\!=} \mathrm{id}_{G(F(x))}$$

- for $x, y, z : \mathrm{Ob}\,\mathcal{C}$, and $f : \mathrm{Hom}_{\mathcal{C}}(x, y), g : \mathrm{Hom}_{\mathcal{C}}(y, z)$,

$$\xi_{G\circ F,g,f} :\equiv \mathrm{ap}_G(\xi_{F,g,f}) \cdot \xi_{G,F(g),F(f)},$$

i.e. the concatenation of

$$G(F(g \circ f)) \overset{\mathrm{ap}_G(\xi_{F,g,f})}{=\!=\!=\!=} G(\mathrm{id}_{F(x)}) \overset{\xi_{G,F(g),F(f)}}{=\!=\!=\!=} \mathrm{id}_{G(F(x))}$$

The identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined as the identity on objects and morphisms, and using reflexivity for ι and ξ.

Proposition 5.1.15. *Fixing a universe of types $\mathcal{U}$, the wild precategories in $\mathcal{U}$ form a wild precategory in which the morphisms are the functors, with composition and identities as defined above.*

Proof. We only need to check that the composition of functors is associative and unital, which follows from the general fact that whenever $f : X \to Y$ and $g : Y \to Z$ are maps between types, we have

$$\mathrm{ap}_{g \circ f} = \mathrm{ap}_g \circ \mathrm{ap}_f .$$

This fact itself follows easily from function extensionality and path induction. $\square$

Remark 5.1.16. Notice that the above definition of functor is not required to preserve all the structure of wild precategories, a less wild notion of functor could for instance require the data of the following higher coherences, witnessing a compatibility with the unitors λ, ρ and the associator α:

$$\xi_{F,\mathrm{id}_y,f} \cdot \mathrm{ap}_{-\circ F(f)}\left(\iota_{F,y}\right) \cdot \lambda_{F(f)} = \mathrm{ap}_F\left(\lambda_f\right)$$

$$\xi_{F,f,\mathrm{id}_x} \cdot \mathrm{ap}_{F(f)\circ -}\left(\iota_{F,x}\right) \cdot \rho_{F(f)} = \mathrm{ap}_F\left(\rho_f\right) \tag{5.1}$$

$$\xi_{F,h\circ g,f} \cdot \mathrm{ap}_{-\circ F(f)}\left(\xi_{F,h,g}\right) \cdot \alpha_{F(h),F(g),F(f)} = \mathrm{ap}_F\left(\alpha_{h,g,f}\right) \cdot \xi_{F,h,g\circ f} \cdot \mathrm{ap}_{F(h)\circ -}\left(\xi_{F,g,f}\right)$$

or, in diagrammatic form:

$$
\begin{array}{ccc}
F(\mathrm{id}_y \circ f) & \xrightarrow{\mathrm{ap}_F(\lambda_f)} & F(f) \\
{\scriptstyle \xi_{F,\mathrm{id}_y,f}}\downarrow & & \uparrow{\scriptstyle \lambda_{F(f)}} \\
F(\mathrm{id}_y) \circ F(f) & \xrightarrow[\mathrm{ap}_{-\circ F(f)}(\iota_{F,y})]{} & \mathrm{id}_{F(y)} \circ F(f)
\end{array}
\qquad
\begin{array}{ccc}
F(f \circ \mathrm{id}_x) & \xrightarrow{\mathrm{ap}_F(\rho_f)} & F(f) \\
{\scriptstyle \xi_{F,f,\mathrm{id}_x}}\downarrow & & \uparrow{\scriptstyle \rho_{F(f)}} \\
F(f) \circ F(\mathrm{id}_x) & \xrightarrow[\mathrm{ap}_{F(f)\circ -}(\iota_{F,x})]{} & F(f) \circ \mathrm{id}_{F(x)}
\end{array}
$$

$$
\begin{array}{ccc}
F((h \circ g) \circ f) \xrightarrow{\xi_{F,h\circ g,f}} F(h \circ g) \circ F(f) \xrightarrow{\mathrm{ap}_{-\circ F(f)}(\xi_{F,h,g})} (F(h) \circ F(g)) \circ F(f) \\
{\scriptstyle \mathrm{ap}_F(\alpha_{h,g,f})}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \alpha_{F(h),F(g),F(f)}} \\
F(h \circ (g \circ f)) \xrightarrow[\xi_{F,h,g\circ f}]{} F(h) \circ F(g \circ f) \xrightarrow[\mathrm{ap}_{F(h)\circ -}(\xi_{F,g,f})]{} F(h) \circ (F(g) \circ F(f))
\end{array}
$$

Notably, it can be shown that the wild precategory of wild precategories and functors is not univalent, while the wild precategory of wild precategories and functors with the added coherences above is univalent.

Also note that when the Hom-types $\mathrm{Hom}_{\mathcal{D}}(x,y)$ are sets, these coherences are always satisfied in a unique way.

Definition 5.1.17. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between wild precategories. A *natural transformation* $\varphi : F \Rightarrow G$ is a pair (φ, p) where:

- for every $x : \mathrm{Ob}\,\mathcal{C}$, a morphism $\varphi_x : F(x) \to G(x)$ in $\mathcal{D}$,

- for every morphism $f : x \to y$ in $\mathcal{C}$,

$$\varphi_f : G(f) \circ \varphi_x = \varphi_y \circ F(f)$$

i.e., in diagrammatic form, a commutative square:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{\varphi_x} & G(x) \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(y) & \xrightarrow[\varphi_y]{} & G(y)
\end{array}
$$

It is said to be a *natural isomorphism* if for every $x : \mathrm{Ob}\,\mathcal{C}$, the morphism $\varphi_x : F(f) \to G(x)$ is an isomorphism in $\mathcal{D}$.

Definition 5.1.18 (Vertical composition)**.** Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors between wild precategories, and $\varphi : F \Rightarrow G$, $\psi : G \Rightarrow H$ be natural transformations. The *vertical composition* $\psi \circ \varphi : F \Rightarrow H$ of φ and ψ is the natural transformation defined on objects by

$$(\psi \circ \varphi)_x :\equiv (\psi_x \circ \varphi_x)$$

and whose naturality squares are given by pasting the squares of φ and ψ:

$$
\begin{array}{ccccc}
F(x) & \xrightarrow{\;\varphi_x\;} & G(x) & \xrightarrow{\;\psi_x\;} & H(x) \\
\big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} & & \big\downarrow{\scriptstyle H(f)} \\
F(y) & \xrightarrow[\;\varphi_y\;]{} & G(y) & \xrightarrow[\;\psi_y\;]{} & H(y)
\end{array}
$$

More precisely, if

$$p : G(f) \circ \varphi_x = \varphi_y \circ F(f),$$
$$q : H(f) \circ \psi_x = \psi_y \circ G(f),$$

we have

$$
\begin{aligned}
H(f) \circ (\psi_x \circ \varphi_x) = (H(f) \circ \psi_x) \circ \varphi_x & \quad (\text{ by } \alpha^{-1}_{\varphi_x, \psi_x, H(f)}) \\
= (\psi_y \circ G(f)) \circ \varphi_x & \quad (\text{ by } \mathrm{ap}_{-\circ\varphi_x}(q)) \\
= \psi_y \circ (G(f) \circ \varphi_x) & \quad (\text{ by } \alpha_{\psi_y, G(f), \varphi_x}) \\
= \psi_y \circ (\varphi_y \circ F(f)) & \quad (\text{ by } \mathrm{ap}_{\psi_y \circ -}(p)) \\
= (\psi_y \circ \varphi_y) \circ F(f) & \quad (\text{ by } \alpha^{-1}_{\psi_y, \varphi_y, F(f)})
\end{aligned}
\tag{5.2}
$$

Having defined functors, natural transformations and vertical composition, one might want to define a wild precategory with functors as objects and natural transformations as morphisms. However this cannot work: the operation of vertical composition is in general not associative!

To understand why, consider the following diagram of wild precategories, functors and natural transformations:

$$
\mathcal{C} \;
\begin{array}{c}
\overset{F}{\underset{\Downarrow(\varphi, p)}{}} \\
\overset{G}{\underset{\Downarrow(\psi, q)}{}} \\
\overset{H}{\underset{\Downarrow(\theta, r)}{}} \\
K
\end{array}
\; \mathcal{D}
$$
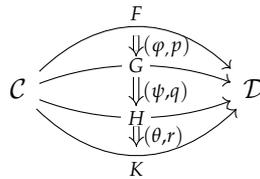
Writing $q * p$ for the path witnessing the composition of naturality squares as defined in (5.2), we would like to prove that

$$((\theta \circ \psi) \circ \varphi, (r * q) * p) \;=\; (\theta \circ (\psi \circ \varphi), r * (q * p))$$

Using the characterization of paths in dependent sums (fact 4.5.6), we see that this amounts to finding for every $x : \mathrm{Ob}\,\mathcal{C}$ a path

$$u_x : (\theta_x \circ \psi_x) \circ \varphi_x = \theta_x \circ (\psi_x \circ \varphi_x)$$

together with, for every $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$, a path witnessing the commutativity of the following square of paths:

$$
\begin{array}{ccc}
K(f) \circ ((\theta_x \circ \psi_x) \circ \phi_x) & \xrightarrow{\mathrm{ap}_{K(f)\circ -} (u_x)} & K(f) \circ (\theta_x \circ (\psi_x \circ \phi_x)) \\
{\scriptstyle (r*q)*p} \downarrow & & \downarrow {\scriptstyle r*(q*p)} \\
((\theta_x \circ \psi_x) \circ \phi_x) \circ F(f) & \xrightarrow[\mathrm{ap}_{-\circ F(f)} (u_x)]{} & (\theta_x \circ (\psi_x \circ \phi_x)) \circ F(f)
\end{array}
$$

There is really one conceivable candidate for u_x, namely

$$
u_x :\equiv \alpha_{\theta_x, \psi_x, \varphi_x}.
$$

Unfolding the definitions of $(r * q) * p$ and $r * (q * p)$, we see that the above square has no reason to be commutative without additional assumptions on the associator α, for instance Mac Lane's pentagon identity. This kind of issue, together with remark 5.1.16, are the typical drawbacks to be encountered when working with wild categories. To be able to still work with natural transformations, we will state all our results using a weaker notion of equality for them:

Definition 5.1.19. Let $\varphi, \psi : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ be natural transformations between functors between wild precategories. We write

$$
(\varphi \sim \psi) :\equiv \prod_{x:\mathrm{Ob}\,\mathcal{C}} (\varphi_x = \psi_x)
$$

for the type of *wild equalities* between φ and ψ.

When $\mathcal{D}$ has set-level Hom-types, it is straightforward to check that

$$
(\varphi \sim \psi) \simeq (\varphi = \psi),
$$

but that is of course not the case for more general $\mathcal{D}$.

Using associativity in the target category $\mathcal{D}$, we immediately have the following:

Proposition 5.1.20. *Given functors $F, G, H, K : \mathcal{C} \to \mathcal{D}$ between wild precategories, and natural transformations $\varphi : F \Rightarrow G$, $\psi : G \Rightarrow H$, $\theta : H \Rightarrow K$, we have*

$$
(\theta \circ \psi) \circ \varphi \sim \theta \circ (\psi \circ \varphi).
$$

Definition 5.1.21 (Whiskering)**.** Consider a diagram of wild precategories, functors and natural transformations as follows:

$$
\mathcal{B} \xrightarrow{\ K\ } \mathcal{C} \overset{F}{\underset{G}{\Downarrow \varphi}} \mathcal{D} \xrightarrow{\ H\ } \mathcal{E}
$$

The *left whiskering* of φ by H is the natural transformation

$$
H * \phi : H \circ F \Rightarrow H \circ G
$$

defined for $x, y : \mathrm{Ob}\,\mathcal{C}$ and $f : \mathrm{Hom}_{\mathcal{C}}(x, y)$ by

$$
(H * \phi)_x :\equiv H(\phi_x)
$$
$$
(H * \phi)_f :\equiv \xi_H^{-1} \cdot H(\phi_f) \cdot \xi_H
$$

The *right whiskering* of φ by K is the natural transformation

$$\phi * K : F \circ K \Rightarrow G \circ K$$

defined for $x, y : \mathrm{Ob}\,\mathcal{B}$ and $f : \mathrm{Hom}_{\mathcal{B}}(x, y)$ by

$$(\phi * K)_x :\equiv \phi_{K(x)}$$
$$(\phi * K)_f :\equiv \phi_{K(f)} (H * \phi)_x :\equiv H(\phi_x)$$
$$(H * \phi)_f :\equiv \xi_H^{-1} \cdot H(\phi_f) \cdot \xi_H$$

Proposition 5.1.22. *Consider a diagram of wild precategories, functors and natural transformations as follows:*

$$\mathcal{C} \overset{F}{\underset{F'}{\Downarrow \varphi}} \mathcal{D} \overset{G}{\underset{G'}{\Downarrow \psi}} \mathcal{E}$$

We have a wild equality of natural transformations

$$(\psi * F') \circ (G * \varphi) \ \sim \ (G' * \varphi) \circ (\psi * F).$$

Remark 5.1.23. In classical category theory, the above wild equality is an actual equality between natural transformations, so that we can define the *horizontal composition $\psi * \varphi$* using either of the two above formulae. In the *wild* setting we must be more careful, choosing one of the two possible definitions once and for all. In this thesis we won't rely on explicit horizontal compositions anywhere, instead relying directly on whiskering wherever needed, so we leave this choice open to future work.

5.2 Wild structures on wild categories

Armed with suitable notions of categories, functors and natural transformations, we can define wild variants of traditional structures on categories and functors: notably with the goal of interpreting linear logic, we need closed symmetric monoidal structures and (co)monads. We first delve into symmetric monoidal structures.

Definition 5.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be wild precategories. Their *cartesian product $\mathcal{C} \times \mathcal{D}$* is the wild precategory with

- $\mathrm{Ob}\,(\mathcal{C} \times \mathcal{D}) :\equiv (\mathrm{Ob}\,\mathcal{C}) \times (\mathrm{Ob}\,\mathcal{D})$,

- $\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}((x, x'), (y, y')) :\equiv \mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{D}}(x', y')$,

- identities and composition defined pairwise,

- unitors and associators defined pairwise using the characterization of paths in cartesian products (fact 4.5.5).

Definition 5.2.2. Let $\mathcal{C}$ be a wild precategory. A *symmetric monoidal structure* on $\mathcal{C}$ is the data of:

1. a functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*,

2. a distinguished object $\mathbf{1} : \mathrm{Ob}\,\mathcal{C}$ called the *unit*,

3. natural isomorphisms $\lambda_x : \mathbf{1} \otimes x \to x$, $\rho_x : x \otimes \mathbf{1} \to x$ called the *left and right unitors*,

4. a natural isomorphism $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the *associator*,

5. a natural isomorphism $\gamma_{x,y} : x \otimes y \to y \otimes x$ called the *braiding*,

6. witnesses of commutativity for the following diagrams for every $x, y, z, t : \mathrm{Ob}\,\mathcal{C}$:

$$
\begin{array}{ccc}
& (x \otimes y) \otimes (z \otimes t) & \\
\overset{\alpha_{x \otimes y, z, t}}{\nearrow} & & \overset{\alpha_{x, y, z \otimes t}}{\searrow} \\
((x \otimes y) \otimes z) \otimes t & & x \otimes (y \otimes (z \otimes t)) \\
\downarrow{\scriptstyle \alpha_{x,y,z} \otimes \mathrm{id}_t} & & \uparrow{\scriptstyle \mathrm{id}_x \otimes \alpha_{y,z,t}} \\
(x \otimes (y \otimes z)) \otimes t & \xrightarrow{\;\;\alpha_{x, y \otimes z, t}\;\;} & x \otimes ((y \otimes z) \otimes t)
\end{array}
$$

$$
\begin{array}{ccc}
(x \otimes \mathbf{1}) \otimes y & \xrightarrow{\;\;\alpha_{x,\mathbf{1},y}\;\;} & x \otimes (\mathbf{1} \otimes y) \\
{\scriptstyle \rho_x \otimes \mathrm{id}_y}\searrow & & \swarrow{\scriptstyle \mathrm{id}_x \otimes \lambda_y} \\
& x \otimes y &
\end{array}
$$

$$
\begin{array}{ccccc}
(x \otimes y) \otimes z & \xrightarrow{\alpha_{x,y,z}} & x \otimes (y \otimes z) & \xrightarrow{\gamma_{x, y \otimes z}} & (y \otimes z) \otimes x \\
{\scriptstyle \gamma_{x,y} \otimes \mathrm{id}_z}\downarrow & & & & \downarrow{\scriptstyle \alpha_{y,z,x}} \\
(y \otimes x) \otimes z & \xrightarrow[\alpha_{y,x,z}]{} & y \otimes (x \otimes z) & \xrightarrow[\mathrm{id}_y \otimes \gamma_{x,z}]{} & y \otimes (z \otimes x)
\end{array}
$$

$$
x \otimes y \xrightarrow{\gamma_{x,y}} y \otimes x \xrightarrow{\gamma_{y,x}} x \otimes y
$$
$$
\underset{\mathrm{id}_{x \otimes y}}{}
$$

The equalities for these four diagrams are respectively called the *pentagon identity*, the *triangle identity*, the *hexagon identity* and the *symmetry identity* of the symmetric monoidal structure.

A wild precategory $\mathcal{C}$ equipped with a symmetric monoidal structure is called a symmetric monoidal wild precategory.

Remark 5.2.3. The names "unitor" and "associator" might cause some confusion as they are already used in the definition of wild precategories. However in practice it will be clear from the context which kind of unitor/associator we are talking about. Notice in particular the difference between their underlying types: the associator for composition in $\mathcal{C}$ is an equality between morphisms, while the associator for a symmetric monoidal structure is a natural isomorphism.

Remark 5.2.4. This definition is the same as the usual definition of symmetric monoidal category, only here we are working with arbitrary *wild* variants of categories, functors and natural transformations. Notably the commutativity of the last four diagrams does not entail any equality of natural transformations, only wild equalities.

Having defined symmetric monoidal structures, we can now define internal hom objects:

Definition 5.2.5. Let $\mathcal{C}$ be a symmetric monoidal wild precategory. Let $x, y : \mathrm{Ob}\,\mathcal{C}$. An *internal hom* from x to y in $\mathcal{C}$ is an object $z : \mathrm{Ob}\,\mathcal{C}$, together with a morphism

$$
\mathrm{ev} : z \otimes x \to y
$$

called the *evaluation morphism*, such that for every object $t : \mathrm{Ob}\,\mathcal{C}$ equipped with a morphism $e : t \otimes x \to y$, there exists a unique morphism $u : t \to z$ such that the following triangle commutes:

$$t \otimes x \xrightarrow{u \otimes \mathrm{id}_x} z \otimes x$$
$$\searrow_{e} \qquad \downarrow_{\mathrm{ev}}$$
$$y$$

Said in a more type-theoretic manner, the *type of internal homs* from x to y in $\mathcal{C}$ is the type

$$\sum_{z:\mathrm{Ob}\,\mathcal{C}} \sum_{\mathrm{ev}:z\otimes x\to y} \prod_{t:\mathrm{Ob}\,\mathcal{C}} \prod_{e:t\otimes x\to y} \mathrm{isContr}\left(\sum_{u:t\to z} \mathrm{ev}\circ(u\otimes\mathrm{id}_y) = e \right).$$

Remark 5.2.6. Another classical way of defining internal homs in classical category theory is to say that (z, ev) is an internal hom from x to y if, for every $t : \mathrm{Ob}\,\mathcal{C}$, the map

$$\theta_t : \mathrm{Hom}_{\mathcal{C}}(t, z) \to \mathrm{Hom}_{\mathcal{C}}(t \otimes x, y)$$
$$u \mapsto \mathrm{ev}\circ(u \otimes \mathrm{id}_x)$$

is a bijection. In type theory, we can instead ask this map to be an equivalence, and we also get an equivalent definition of internal homs. To see why this is the case, consider the following:

Given a morphism $e : t \otimes x \to y$ in $\mathcal{C}$, the fiber $\mathrm{fib}_{\theta_t}(e)$ is by definition the type

$$\sum_{u:t\to z} \mathrm{ev}\circ(u \otimes \mathrm{id}_x) = e.$$

The pair (z, ev) being an internal hom precisely means that this type is contractible for every t and e. By fact 4.9.5, this is equivalent to asking for the map θ_t to be an equivalence for every $t : \mathrm{Ob}\,\mathcal{C}$, which gives us the desired conclusion.

Definition 5.2.7. A *closed* symmetric monoidal wild precategory is a symmetric monoidal wild precategory equipped with an internal hom for every pair of objects $x, y : \mathrm{Ob}\,\mathcal{C}$. In a closed symmetric monoidal wild precategory, we write $x \multimap y$ for the underlying object of the specified internal hom from x to y.

Proposition 5.2.8. *Let $\mathcal{C}$ be a symmetric monoidal wild category, i.e. a symmetric monoidal wild precategory which is moreover univalent. Then $\mathcal{C}$ being closed is a proposition.*

More specifically, for every pair of objects $x, y : \mathrm{Ob}\,\mathcal{C}$, the type of internal homs from x to y in $\mathcal{C}$ is a proposition.

Proof. The first assertion follows from the second one by fact 4.7.20. We prove the second assertion: let $x, y : \mathrm{Ob}\,\mathcal{C}$ and consider two internal homs

$$(z, \mathrm{ev}, f), (z', \mathrm{ev}', f') : \sum_{z:\mathrm{Ob}\,\mathcal{C}} \sum_{\mathrm{ev}:z\otimes x\to y} \prod_{t:\mathrm{Ob}\,\mathcal{C}} \prod_{e:t\otimes x\to y} \mathrm{isContr}\left(\sum_{u:t\to z} \mathrm{ev}\circ(u\otimes\mathrm{id}_y) = e \right)$$

We want to show that $(z, \mathrm{ev}, f) = (z', \mathrm{ev}', f')$. Being contractible is a proposition (fact 4.7.12) and propositions are stable under dependent products (fact 4.7.20), so by fact 4.7.13 we only have to construct a path $(z, \mathrm{ev}) = (z', \mathrm{ev}')$. Looking at the centers of contraction of $f(z', \mathrm{ev}')$ and $f'(z, \mathrm{ev})$, we get maps $u : z \to z'$ and $v : z' \to z$ together with witnesses of commutativity of the following triangles:

$$z \otimes x \xrightarrow{u \otimes \mathrm{id}_x} z' \otimes x \qquad\qquad z' \otimes x \xrightarrow{v \otimes \mathrm{id}_x} z \otimes x$$
$$\searrow_{\mathrm{ev}} \;\; \downarrow_{\mathrm{ev}'} \qquad\qquad\qquad \downarrow_{\mathrm{ev}'} \;\; \swarrow_{\mathrm{ev}} \qquad\qquad (5.3)$$
$$y \qquad\qquad\qquad\qquad\qquad y$$

In particular, pasting these triangles together, we get a commutative diagram

$$z \otimes x \xrightarrow{u \otimes \mathrm{id}_x} z' \otimes x \xrightarrow{v \otimes \mathrm{id}_x} z \otimes x$$

$$\mathrm{ev} \searrow \quad y \quad \swarrow \mathrm{ev}$$

By functoriality of $\otimes$, this means we have an equality

$$p : \mathrm{ev} \circ ((v \circ u) \otimes \mathrm{id}_x) = \mathrm{ev}.$$

Independently, we can also use the functoriality of $\otimes$ to get an equality

$$q : \mathrm{ev} \circ (\mathrm{id}_z \otimes \mathrm{id}_x) = \mathrm{ev}.$$

In other words, we have two pairs $(v \circ u, p)$ and (id_z, q) in the type

$$\sum_{u : z \to z} \mathrm{ev} \circ (u \otimes \mathrm{id}_x) = \mathrm{ev},$$

which is contractible by $f(z, \mathrm{ev})$, so $(v \circ u, p) = (\mathrm{id}_z, q)$. In particular, we get an equality $v \circ u = \mathrm{id}_z$. A symmetric argument shows that $u \circ v = \mathrm{id}_{z'}$, so u and v determine an isomorphism $(u, \dots) : \mathrm{Iso}_{\mathcal{C}}(z, z')$.

So far we followed the classical reasoning used to show that objects in a category satisfying universal properties are unique up to unique isomorphism. But $\mathcal{C}$ being univalent gets us further: by univalence of $\mathcal{C}$, the isomorphism u gives us an equality $\mathrm{ua}(u) : x = y$. By the characterization of paths in dependent sums (fact 4.5.6), we are left with proving that $^{\mathrm{ua}(u)}(\mathrm{ev}) = \mathrm{ev}'$. We can show by path induction that for every $r : z = z'$, there is an equivalence

$$(r_*(\mathrm{ev}) = \mathrm{ev}') \simeq (\mathrm{ev}' \circ (\mathrm{idtoiso}(p) \otimes \mathrm{id}_x) = \mathrm{ev}).$$

In particular,

$$(\mathrm{ev}' \circ (\mathrm{idtoiso}(\mathrm{ua}(u)) \otimes \mathrm{id}_x) = \mathrm{ev}) \to \mathrm{ua}(u)_*(\mathrm{ev}) = \mathrm{ev}',$$

but ua is by definition an inverse to idtoiso, so we are left with proving that

$$\mathrm{ev}' \circ (u \otimes \mathrm{id}_x) = \mathrm{ev},$$

which is true by definition of u (see the left diagram 5.3). $\qquad\square$

Remark 5.2.9. While our definition of symmetric monoidal structure is *wild* in that an actual symmetric monoidal ∞-category is required to have higher coherences that we simply discard here, our definition of being closed, being propositional, already satisfies all higher coherences we could ask for. This is similar to the fact that we could get a homotopy-coherent definition of equivalences by defining isEquiv in a way that isEquiv (f) is a proposition.

A concrete consequence of this is that we can use homotopy type theory to prove properties about actual ∞-categories: the definition of internal homs only relies on the low-dimensional data in an ∞-category. Given a symmetric monoidal ∞-category $\mathcal{C}$ for which we can define in HoTT its underlying symmetric monoidal wild category $|\mathcal{C}|$, a proof that $|\mathcal{C}|$ is closed constitutes a proof that the original symmetric monoidal ∞-category $\mathcal{C}$ is closed. Making this more formal would require delving into the (higher) categorical semantics of homotopy type theory, something we will not undertake here, and we redirect the interested reader to Emily Riehl's survey [Rie24].

We now turn to functors between symmetric monoidal wild precategories. Again, the definition is the usual one from classical category theory, without asking for higher coherences.

Definition 5.2.10. Let $(\mathcal{C}, \otimes, \mathbf{1})$ and $(\mathcal{D}, \odot, \mathbb{I})$ be symmetric monoidal wild precategories. Write $\lambda^{\mathcal{C}}, \rho^{\mathcal{C}}, \alpha^{\mathcal{C}}, \gamma^{\mathcal{C}}$ respectively for the left unitor, right unitor, associator and braiding of the symmetric monoidal structure on $\mathcal{C}$, and similarly $\lambda^{\mathcal{D}}, \rho^{\mathcal{D}}, \alpha^{\mathcal{D}}, \gamma^{\mathcal{D}}$ for the corresponding natural transformations on $\mathcal{D}$.

A *lax symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a morphism $m^0 : \mathbb{I} \to F(\mathbf{1})$ and a natural transformation

$$m^2 : F(-) \odot F(-) \to F(- \otimes -),$$

together witnesses of commutativity for the following diagrams, exhibiting compatibilities between m^0, m^2 and the symmetric monoidal structure:

$$
\begin{array}{ccc}
(Fx \odot Fy) \odot Fz & \xrightarrow{\alpha^{\mathcal{D}}_{Fx,Fy,Fz}} & Fx \odot (Fy \odot Fz) \\
{\scriptstyle m^2_{x,y} \otimes \mathrm{id}_{Fz}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{Fx} \odot m^2_{y,z}} \\
F(x \otimes y) \odot Fz & (1) \quad Fx \odot F(y \otimes z) \\
{\scriptstyle m^2_{x \otimes y, z}} \downarrow & & \downarrow {\scriptstyle m^2_{x, y \otimes z}} \\
F((x \otimes y) \otimes z) & \xrightarrow[F(\alpha^{\mathcal{C}}_{x,y,z})]{} & F(x \otimes (y \otimes z))
\end{array}
$$

$$
\begin{array}{ccc}
Fx \odot \mathbb{I} & \xrightarrow{\rho^{\mathcal{D}}_{Fx}} & Fx \\
{\scriptstyle \mathrm{id}_{Fx} \odot m^0} \downarrow & (2) & \uparrow {\scriptstyle F(\rho^{\mathcal{C}}_x)} \\
Fx \odot F\mathbf{1} & \xrightarrow[m^2_{x,1}]{} & F(x \otimes \mathbf{1})
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{I} \odot Fx & \xrightarrow{\lambda^{\mathcal{D}}_{Fx}} & Fx \\
{\scriptstyle m^0 \odot \mathrm{id}_{Fx}} \downarrow & (3) & \uparrow {\scriptstyle F(\lambda^{\mathcal{C}}_x)} \\
F\mathbf{1} \odot Fx & \xrightarrow[m^2_{x,1}]{} & F(\mathbf{1} \otimes x)
\end{array}
$$

$$
\begin{array}{ccc}
Fx \odot Fy & \xrightarrow{\gamma^{\mathcal{D}}_{Fx,Fy}} & Fy \odot Fx \\
{\scriptstyle m^2_{x,y}} \downarrow & (4) & \downarrow {\scriptstyle m^2_{y,x}} \\
F(x \otimes y) & \xrightarrow[F(\gamma^{\mathcal{C}}_{x,y})]{} & F(y \otimes x)
\end{array}
$$

The functor is moreover said to be strongly symmetric monoidal, or simply *symmetric monoidal*, when m^0 is an isomorphism and m^2 is a natural isomorphism.

Finally, we also state the definition of monads in the wild categorical context:

Definition 5.2.11. Let $\mathcal{C}$ be a wild precategory, and $F : \mathcal{C} \to \mathcal{C}$ a functor. A *monad structure* on F is the data of natural transformations

$$\mu : F \circ F \to F \tag{5.4}$$

$$\eta : \mathrm{id}_{\mathcal{C}} \to F \tag{5.5}$$

together with wild equalities

$$\mu \circ (\eta * F) \sim \mathrm{id}_F$$
$$\mu \circ (F * \eta) \sim \mathrm{id}_F$$
$$\mu \circ (F * \mu) \sim \mu \circ (\mu * F)$$

or, in diagrammatic form, for every $x : \mathrm{Ob}\,\mathcal{C}$, equalities:

$$
\begin{array}{ccc}
Fx \xrightarrow{\ F(\eta_x)\ } FFx & \qquad & Fx \xrightarrow{\ \eta_{Fx}\ } FFx \\
\searrow_{\mathrm{id}_{Fx}} \quad \downarrow_{\mu_x} & & \searrow_{\mathrm{id}_{Fx}} \quad \downarrow_{\mu_x} \\
F & & Fx
\end{array}
$$

$$
\begin{array}{ccc}
FFFx & \xrightarrow{\ \mu_{Fx}\ } & FFx \\
\downarrow_{F(\mu_x)} & & \downarrow_{\mu_x} \\
FFx & \xrightarrow[\ \mu_x\ ]{} & Fx
\end{array}
$$

Dually, a *comonad structure* on F is a monad structure on $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

5.3 (Co)limits in wild categories

We now turn to the definition of limits and colimits in wild precategories. Limits and colimits are generally defined in great generality for arbitrary diagrams $D : J \to \mathcal{C}$, where J and $\mathcal{C}$ are categories and D is a functor. In our case, since we are working with *wild* functors, trying such an approach would yield badly behaved notions of limits and colimits. Instead, we focus on specific examples of diagrams, for which we can define limits and colimits by hand without having to rely on a general theory.

Definition 5.3.1. Let $\mathcal{C}$ be a wild precategory. A *terminal object* in $\mathcal{C}$ is an object $x : \mathrm{Ob}\,\mathcal{C}$ such that for every $y : \mathrm{Ob}\,\mathcal{C}$, the type $\mathrm{Hom}_{\mathcal{C}}(y, x)$ is contractible. Dually, x is said to be an *initial object* if for every $y : \mathrm{Ob}\,\mathcal{C}$, the type $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is contractible.

Once again, the usual *uniqueness* of morphisms from classical category theory is replaced by *contractibility* in homotopy type theory. Note that just like in the classical case, x is terminal in $\mathcal{C}$ if and only if it is initial in $\mathcal{C}^{\mathrm{op}}$.

Example 5.3.2. In the wild category of the universe (example 5.1.3), given a type X, the following are equivalent:

1. X is terminal,

2. X is contractible,

3. $X \simeq 1$.

Example 5.3.3. In the same setting as the previous example, the following are equivalent:

1. X is initial,

2. X is *empty*, i.e. there is a map $X \to \varnothing$,

3. $X \simeq \varnothing$.

Proposition 5.3.4. *In a wild category $\mathcal{C}$ (i.e. when $\mathcal{C}$ is univalent), the type*

$$
\sum_{x:\mathrm{Ob}\,\mathcal{C}} \prod_{y:\mathrm{Ob}\,\mathcal{C}} \mathrm{isContr}\left(\mathrm{Hom}_{\mathcal{C}}(y, x)\right)
$$

of terminal objects in $\mathcal{C}$ is a proposition.

Proof. The proof is an extremely simplified variant of the proof of proposition 5.2.8. Let $(x, f), (y, g) :$ $\sum_{x:\mathrm{Ob}\,\mathcal{C}} \prod_{y:\mathrm{Ob}\,\mathcal{C}} \mathrm{isContr}\,(\mathrm{Hom}_{\mathcal{C}}(y, x))$. The type $\prod_{y:\mathrm{Ob}\,\mathcal{C}} \mathrm{isContr}\,(\mathrm{Hom}_{\mathcal{C}}(y, x))$ is propositional by fact 4.7.20 and fact 4.7.12, so by fact 4.7.13 we only have to show that $x = y$. From f and g, we get morphisms $u : x \to y$ and $v : y \to x$. By f again, $\mathrm{Hom}_{\mathcal{C}}(x, x)$ is contractible, so $v \circ u = \mathrm{id}_x$. Symmetrically, $u \circ v = \mathrm{id}_y$, so u is an isomorphism, and by univalence of $\mathcal{C}$, we get an equality $\mathrm{ua}\,(u) : x = y$. $\qquad\square$

As a direct corollary of proposition 5.3.4 and proposition 5.1.11, we have:

Corollary 5.3.5. *In a wild category $\mathcal{C}$, the type of initial objects is a proposition.*

Definition 5.3.6 (Cartesian product)**.** Let $\mathcal{C}$ be a wild precategory, and $x, y : \mathrm{Ob}\,\mathcal{C}$. A *cartesian product* — or simply product — of x and y in $\mathcal{C}$ is an object z together with morphisms $\pi_1 : z \to x$, $\pi_2 : z \to y$ such that for every object $t : \mathrm{Ob}\,\mathcal{C}$ and morphisms $f : t \to x$, $g : t \to y$, the there is a unique morphism $h : t \to z$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

In a more type-theoretic fashion, the type of cartesian products of x and y in $\mathcal{C}$ is

$$\sum_{z:\mathrm{Ob}\,\mathcal{C}} \sum_{\pi_1:z\to x} \sum_{\pi_2:z\to y} \prod_{t:\mathrm{Ob}\,\mathcal{C}} \prod_{f:t\to x} \prod_{g:t\to y} \mathrm{isContr}\left(\sum_{h:t\to z} (\pi_1 \circ h = f) \times (\pi_2 \circ h = g) \right)$$

Definition 5.3.7 (Coproduct)**.** A *coproduct* of x and y in $\mathcal{C}$ is a cartesian product of x and y in $\mathcal{C}^{\mathrm{op}}$.

Proposition 5.3.8. *In a wild category $\mathcal{C}$, the type of cartesian products of x and y is a proposition.*

Proof. The proof follows the same lines as the proofs of propositions 5.2.8 and 5.3.4. $\qquad\square$

Example 5.3.9. In the wild category of the universe $\mathcal{U}$, the cartesian products of types X and Y is precisely the product type $X \times Y$, together with its usual projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$.

Example 5.3.10. Again in $\mathcal{U}$, the coproduct of types X and Y is precisely their coproduct type $X \sqcup Y$, together with its usual injections $\iota_1 : X \to X \sqcup Y$ and $\iota_2 : Y \to X \sqcup Y$.

Definition 5.3.11. We say that a wild precategory $\mathcal{C}$ has *finite products* (or is *cartesian*) if it has a terminal object 1 and a cartesian products $x \times y$ for every pair of objects $x, y : \mathrm{Ob}\,\mathcal{C}$.

Definition 5.3.12 (Cartesian monoidal structure)**.** Let $\mathcal{C}$ be a wild precategory with finite products $(1, - \times -)$. There is a symmetric monoidal structure on $\mathcal{C}$, called the *cartesian symmetric monoidal structure*, in which $\mathbf{1} :\equiv 1$ and $\otimes :\equiv \times$. If $\mathcal{C}$ is closed with respect to this structure, we say it is *cartesian closed*.

Remark 5.3.13. When $\mathcal{C}$ is univalent, the cartesian symmetric monoidal structure, is uniquely determined if it exists (by propositions 5.3.4 and 5.3.8).

Definition 5.3.14. Let $\mathcal{C}$ be a wild precategory, and consider a cospan diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{\ f\ } & z \end{array}$$

A *pullback* of (x, y, z, f, g) in $\mathcal{C}$ is an object $u : \mathrm{Ob}\,\mathcal{C}$ together with morphisms $\pi_1 : u \to x$, $\pi_2 : u \to y$, and a path $p : f \circ \pi_1 = g \circ \pi_2$ such that, for every $t : \mathrm{Ob}\,\mathcal{C}$, every $i : t \to x$, $j : t \to y$ and $q : f \circ i = g \circ j$, there is a unique morphism $h : t \to u$ equipped with paths

$$r : \pi_1 \circ h = i$$
$$s : \pi_2 \circ h = j$$

and a higher path between composite paths:

This last equality can be written more formally as a commutativity of the following concatenation of paths:

$$f \circ i \xrightarrow{\mathrm{ap}_{f \circ -}\left(r^{-1}\right)} f \circ (\pi_1 \circ h) \xrightarrow{\alpha^{-1}_{f,\pi_1,h}} (f \circ \pi_1) \circ h \xrightarrow{\mathrm{ap}_{- \circ h}(p)} (g \circ \pi_2) \circ h \xrightarrow{\alpha_{g,\pi_2,h}} g \circ (\pi_2 \circ h) \xrightarrow{\mathrm{ap}_{g \circ -}(s)} g \circ j$$

$$\underset{q}{\underbrace{}}$$

Written out in full, the type of pullbacks of (x, y, z, f, g) is the following:

$$\sum_{u:\mathrm{Ob}\,\mathcal{C}} \sum_{\pi_1:u\to x} \sum_{\pi_2:u\to y} \sum_{p:f\circ\pi_1=g\circ\pi_2}$$

$$\prod_{t:\mathrm{Ob}\,\mathcal{C}} \prod_{i:t\to x} \prod_{j:t\to y} \prod_{q:f\circ i=g\circ j}$$

$$\mathrm{isContr}\Big(\sum_{h:t\to u} \sum_{r:\pi_1\circ h=i} \sum_{s:\pi_2\circ h=j}$$

$$\big(\mathrm{ap}_{f\circ-}\left(r^{-1}\right) \cdot \alpha^{-1}_{f,\pi_1,h} \cdot \mathrm{ap}_{-\circ h}(p) \cdot \alpha_{g,\pi_2,h} \cdot \mathrm{ap}_{g\circ-}(s)\big) = q\Big)$$

Just like for internal homs, terminal objects and cartesian products, we have the following unicity result for pullbacks:

Proposition 5.3.15. *When $\mathcal{C}$ is a wild category, the above type of pullbacks is a proposition.*

Example 5.3.16 (Pullbacks in types). Let X, Y, Z be types, and $f : X \to Z$, $g : Y \to Z$ be maps of types. The pullback of (X, Y, Z, f, g) in the wild category $\mathcal{U}$ is the type

$$X \times_Z Y :\equiv \sum_{x:X} \sum_{y:Y} f(x) = g(y),$$

with $\pi_1 : X \times_Z Y \to X$ and $\pi_2 : X \times_Z Y \to Y$ defined as the actual first and second projections from the dependent sum type. The equality $p : f \circ \pi_1 = g \circ \pi_2$ is given by function extensionality applied to the dependent map

$$\prod_{t:X\times_Z Y} \pi_1(t) = \pi_2(t)$$

$$(x, y, p) \mapsto p$$

This definition of the pullback $X \times_Z Y$ is to be compared with the definition of pullbacks of sets in set theory:

$$X \times_Z Y := \{(x,y) \in X \times Y \mid f(x) = g(x)\}.$$

In set theory, we are restricting the set $X \times Y$ along the proposition "$f(x) = g(x)$". In homotopy type theory on the other hand, the type $f(x) = g(x)$ may not be a proposition, and the pullback does not generally embed into the cartesian product.

Example 5.3.17. Let X be a type, and $x : X$ an element of X. The element x determines a map $\mathrm{ind}_1(X, x) : 1 \to X$. The pullback of this map with itself is the type

$$\sum_{a:1} \sum_{b:1} \mathrm{ind}_1(X, x, a) =_X \mathrm{ind}_1(X, x, b),$$

which is equivalent to $x =_X x$, or in other words the loop space $\Omega(X, x)$ of (X, x).

In particular, when $X :\equiv \mathbb{S}^1$, recall from section 4.10 that $\Omega(\mathbb{S}^1, \text{base}) \simeq \mathbb{Z}$, and there is clearly no embedding $\mathbb{Z} \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1$.

Remark 5.3.18. In wild precategories, the definition of terminal objects only makes use of the types $\mathrm{Hom}_{\mathcal{C}}(x, y)$, the definition of cartesian products also relies on the composition operation, and the definition of pullbacks additionally refers to the associator α. All of these are examples of *limits* indexed by different diagrams. The situation is summed up in the following table:

Limit	Diagram shape	Diagram "dimension"
Terminal object	$\varnothing$	-1 (no object)
Cartesian product	$\bullet \qquad \bullet$	0 (only objects)
Pullback	$\begin{array}{c} \bullet \\ \downarrow \\ \bullet \longrightarrow \bullet \end{array}$	1 (objects and morphisms)

As we can see, the higher the dimension of the diagram, the higher dimensional the coherences will be needed to define limits indexed by such diagrams. Going only as far as associators and unitors, we can only define limits indexed by 1-dimensional diagrams, or in other words, graphs.

Following the same reasoning as in remark 5.2.6, it is possible to give the following alternative characterizations for terminal objects, cartesian products and pullbacks:

Proposition 5.3.19. *An object $x : \mathrm{Ob}\,\mathcal{C}$ is terminal if and only if for every $y : \mathrm{Ob}\,\mathcal{C}$, the canonical map*

$$\mathrm{Hom}_{\mathcal{C}}(y, x) \to 1$$
$$h \mapsto \star$$

is an equivalence.

Proposition 5.3.20. *The tuple (z, π_1, π_2) is a cartesian product of x and y in $\mathcal{C}$ if and only if for every $t : \mathrm{Ob}\,\mathcal{C}$, the canonical map*

$$\mathrm{Hom}_{\mathcal{C}}(t, z) \to \mathrm{Hom}_{\mathcal{C}}(t, x) \times \mathrm{Hom}_{\mathcal{C}}(t, y)$$
$$h \mapsto (\pi_1 \circ h, \pi_2 \circ h)$$

is an equivalence.

Proposition 5.3.21. *In a diagram*

$$
\begin{array}{ccc}
u & \xrightarrow{\pi_2} & y \\
{\scriptstyle \pi_1}\downarrow & {\scriptstyle p}\nearrow & \downarrow{\scriptstyle g} \\
x & \xrightarrow{f} & z
\end{array}
$$

the tuple (u, π_1, π_2, p) is a pullback of (x, y, z, f, g) in $\mathcal{C}$ if and only if for every $t : \mathrm{Ob}\,\mathcal{C}$, the canonical map

$$
\mathrm{Hom}_{\mathcal{C}}(t, u) \to \mathrm{Hom}_{\mathcal{C}}(t, x) \times_{\mathrm{Hom}_{\mathcal{C}}(t,z)} \mathrm{Hom}\,(t, y)
$$
$$
h \mapsto (\pi_1 \circ h, \pi_2 \circ h, \alpha_{f,\pi_1,h}^{-1} \cdot \mathrm{ap}_{-\circ h}(p) \cdot \alpha_{g,\pi_2,h})
$$

is an equivalence.

In classical category theory, these results correspond to the way of defining limits in terms of representable functors.

Remark 5.3.22. In remark 5.2.9, we noted that a symmetric monoidal ∞-category being closed can be tested on its underlying symmetric monoidal wild precategory. Similarly, the existence of terminal objects, cartesian products and pullbacks in an ∞-category $\mathcal{C}$ can be tested at the level of its underlying wild precategory.

Even though general limits cannot be defined in wild precategories, it can shown that an ∞-category admits all small limits if and only if it admits small products and pullbacks ([Lur09, Prop 4.4.2.6]). Both these kinds of limits can be defined in wild precategories, so even though we cannot generally test for the existence of *specific* limits in $\mathcal{C}$ on its underlying wild precategory $|\mathcal{C}|$, we can actually test on $|\mathcal{C}|$ whether $\mathcal{C}$ admits *all* small limits at once. We will review the theory of (co)limits in the ∞-categorical setting in section 6.6.

In classical category theory, it is well-known that "limits commute with limits". One particular instance of this — that we will use in section 5.6.2 to show the functoriality of wild categories of internal spans — is that cartesian products commute with pullbacks, in the following sense:

Proposition 5.3.23. *Let $\mathcal{C}$ be a wild precategory, and consider the following diagrams in $\mathcal{C}$:*

$$
\begin{array}{ccc}
 & & y \\
 & & \downarrow{\scriptstyle g} \\
x & \xrightarrow{f} & z
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & y' \\
 & & \downarrow{\scriptstyle g'} \\
x' & \xrightarrow{f'} & z'
\end{array}
$$

Suppose moreover given cartesian products $x \times x'$, $y \times y'$, $z \times z'$, and pullbacks $x \times_z y$, $x' \times_{z'} y'$. Then a pullback $(x \times x') \times_{z \times z'} (y \times y')$ exists if and only if a cartesian product $(x \times_z y) \times (x' \times_{z'} y')$ exists, and they are canonically isomorphic.

Proof. We write $\mathcal{C}(a, b) :\equiv \mathrm{Hom}_{\mathcal{C}}(a, b)$ to simplify the notation in what follows. First, relying on the characterizations of products and pullbacks from propositions 5.3.20 and 5.3.21, we show that for every $t : \mathrm{Ob}\,\mathcal{C}$, there is an equivalence

$$
\mathcal{C}(t, x \times_z y) \times \mathcal{C}(t, x' \times_{z'} y') \simeq \mathcal{C}(t, x \times x') \times_{\mathcal{C}(t,z \times z')} \mathcal{C}(t, y \times y').
$$

Indeed, unfolding the products and pullbacks in $\mathcal{U}$ using examples 5.3.9 and 5.3.16, we have:

$$
\begin{aligned}
& \mathcal{C}(t, x \times_z y) \times \mathcal{C}(t, x' \times_{z'} y') \\
&\simeq (\sum_{i:t \to x} \sum_{j:t \to y} f \circ i = g \circ j) \times (\sum_{i':t \to x'} \sum_{j':t \to y'} f' \circ i' = g' \circ j') \qquad \text{(by proposition 5.3.21)} \\
&\simeq \sum_{i:t \to x} \sum_{i':t \to x'} \sum_{j:t \to y} \sum_{j':t \to y'} (f \circ i = g \circ j) \times (f' \circ i' = g' \circ j') \qquad \text{(reordering)} \\
&\simeq \sum_{I:t \to x \times x'} \sum_{J:t \to y \times y'} ((f \times f') \circ I = (g \times g') \circ J) \qquad \text{(by proposition 5.3.20)} \\
&\simeq \mathcal{C}(t, x \times x') \times_{\mathcal{C}(t,z \times z')} \mathcal{C}(t, y \times y')
\end{aligned}
$$

Now suppose for instance that a product $(x \times_z y) \times (x' \times_{z'} y')$ exists. By composing the equivalence of proposition 5.3.20 with the above equivalence, we get an equivalence

$$
\mathcal{C}(t, (x \times_z y) \times (x' \times_{z'} y')) \simeq \mathcal{C}(t, x \times x') \times_{\mathcal{C}(t,z \times z')} \mathcal{C}(t, y \times y').
$$

Computing the underlying map of this equivalence, we see it is equal to the canonical map of proposition 5.3.21, so the product is also a pullback. The other case where the pullback is assumed to exist first can be concluded through an analogous reasoning. $\qquad \square$

5.4 Truncating wild categories

Given a wild precategory $\mathcal{C}$, one can always recover an actual precategory by truncating its type of objects and its types of morphisms. This is the wild-categorical variant of the homotopy category construction from section 2.3 (definition 2.3.12). As we are interested in models of linear logic, we will in particular see that any *wild model* can be truncated to recover an actual 1-categorical model (theorem 5.5.2).

Definition 5.4.1. Let $\mathcal{C}$ be a wild precategory. We define a precategory $\|\mathcal{C}\|$ in two steps:

1. First observe that the composition operation

$$
\mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)
$$

passes to set-truncation:

$$
\| \mathrm{Hom}_{\mathcal{C}}(y, z) \|_0 \to \| \mathrm{Hom}_{\mathcal{C}}(x, y) \|_0 \to \| \mathrm{Hom}_{\mathcal{C}}(x, z) \|_0,
$$

using the elimination principle for the set-truncation twice. This defines another wild precategory structure on $\mathrm{Ob}\,\mathcal{C}$, where the hom-types are sets.

2. Since hSet is a 1-type (corollary 4.7.27), the map

$$
\begin{aligned}
\mathrm{Ob}\,\mathcal{C} &\to \mathrm{Ob}\,\mathcal{C} \to \mathrm{hSet} \\
x &\mapsto (y \mapsto (\| \mathrm{Hom}_{\mathcal{C}}(x, y) \|_0, \dots))
\end{aligned}
$$

restricts to a map $\| \mathrm{Ob}\,\mathcal{C} \|_1 \to \| \mathrm{Ob}\,\mathcal{C} \|_1 \to \mathrm{hSet}$. This defines a precategory structure on $\| \mathrm{Ob}\,\mathcal{C} \|_1$.

The induced precategory $\|\mathcal{C}\|$ is called the *truncation* of $\mathcal{C}$, or the *homotopy (pre)category* of $\mathcal{C}$.

We now explore what structures and properties of $\mathcal{C}$ can be recovered in its truncation $\|\mathcal{C}\|$.

Proposition 5.4.2. *Let $\mathcal{C}$ be a wild precategory, and $x : \operatorname{Ob}\mathcal{C}$. If x is terminal (resp. initial) in $\mathcal{C}$, then $|x|_1$ is terminal (resp. initial) in $\|\mathcal{C}\|$.*

Proof. Suppose x is terminal (the case of x initial follows by duality). We want to show that for every $a : \|\operatorname{Ob}\mathcal{C}\|_1$, the type $\operatorname{Hom}_{\|\mathcal{C}\|}(a, |x|_1)$ is contractible. Being contractible is a proposition (fact 4.7.12), so by cumulativity of n-types, being contractible is also a 1-type (fact 4.7.7). Hence, using the elimination principle for $\| - \|_1$, it suffices to show that for every $y : \operatorname{Ob}\mathcal{C}$, the type $\operatorname{Hom}_{\|\mathcal{C}\|}(|y|_1, |x|_1)$ is contractible. By definition, this type is judgmentally equal to

$$\| \operatorname{Hom}_{\mathcal{C}}(y, x) \|_0,$$

a set-truncation of a contractible type, so it is itself contractible, which concludes the proof. $\square$

Proposition 5.4.3. *Let $\mathcal{C}$ be a wild precategory, $x, y : \operatorname{Ob}\mathcal{C}$ and $(x \times y, \pi_1, \pi_2)$ a cartesian product of x and y in $\mathcal{C}$. Then $(|x \times y|_1, |\pi_1|_0, |\pi_2|_0)$ is a cartesian product of $|x|_1$ and $|y|_1$ in $\|\mathcal{C}\|$. The same holds dually for coproducts.*

The proof relies on the following fact about truncations and product types:

Fact 5.4.4 ([Uni13, Thm 7.3.8]). *Let X and Y be types, and $n \geq -2$. The canonical map*

$$\|X \times Y\|_n \to \|X\|_n \times \|Y\|_n$$

induced by the elimination principle for $\| - \|_n$ (theorem 4.11.3) applied to the map

$$X \times Y \to \|X\|_n \times \|Y\|_n$$
$$(x, y) \mapsto (|x|_n, |y|_n)$$

is an equivalence of types.

Proof of proposition 5.4.3. Let $(x \times y, \pi_1, \pi_2)$ be a cartesian product of x and y in $\mathcal{C}$. It suffices to show that for every $t : \|\operatorname{Ob}\mathcal{C}\|_1$, the canonical map

$$\operatorname{Hom}_{\|\mathcal{C}\|}(t, \|x \times y\|_1) \to \operatorname{Hom}_{\|\mathcal{C}\|}(t, \|x\|_1) \times \operatorname{Hom}_{\|\mathcal{C}\|}(t, \|y\|_1)$$

is an equivalence. Since being an equivalence is a proposition, it is in particular a 1-type, so we can use the truncation elimination on $t : \|\operatorname{Ob}\mathcal{C}\|_1$ and are left with proving that for every $t : \operatorname{Ob}\mathcal{C}$, the canonical map

$$\operatorname{Hom}_{\|\mathcal{C}\|}(\|t\|_1, \|x \times y\|_1) \to \operatorname{Hom}_{\|\mathcal{C}\|}(\|t\|_1, \|x\|_1) \times \operatorname{Hom}_{\|\mathcal{C}\|}(\|t\|_1, \|y\|_1)$$

is an equivalence. This reduces to

$$\| \operatorname{Hom}_{\mathcal{C}}(t, x \times y) \|_0 \to \| \operatorname{Hom}_{\mathcal{C}}(t, x) \|_0 \times \| \operatorname{Hom}_{\mathcal{C}}(t, y) \|_0.$$

Through the equivalence $\|X\|_0 \times \|Y\|_0 \simeq \|X \times Y\|_0$ of fact 5.4.4, this amounts to proving the canonical map

$$\| \operatorname{Hom}_{\mathcal{C}}(t, x \times y) \|_0 \to \| \operatorname{Hom}_{\mathcal{C}}(t, x) \times \operatorname{Hom}_{\mathcal{C}}(t, y) \|_0$$

is an equivalence. This map is precisely the functorial action of $\| - \|_0$ on the equivalence

$$\operatorname{Hom}_{\mathcal{C}}(t, x \times y) \simeq \operatorname{Hom}_{\mathcal{C}}(t, x) \times \operatorname{Hom}_{\mathcal{C}}(t, y),$$

so it is itself an equivalence, which concludes the proof. $\square$

Proposition 5.4.5. *Let $\mathcal{C}$ be a wild precategory with cartesian products and such that $\|\mathcal{C}\|$ is univalent. Then $\|\mathcal{C}\|$ admits cartesian products.*

Proof. We want to prove that for every $x, y : \|\mathcal{C}\|_1$, x and y admit a cartesian product in $\|\mathcal{C}\|$. Since $\|\mathcal{C}\|$ is univalent, the type of cartesian products of x and y is a proposition (proposition 5.3.8), hence a fortiori a 1-type. Hence we can use the elimination principle for $\| - \|_1$ on $x : \|\operatorname{Ob}\mathcal{C}\|_1$ and $y : \|\operatorname{Ob}\mathcal{C}\|_1$, and are left with proving that for every $x, y : \operatorname{Ob}\mathcal{C}$, $|x|_1$ and $|y|_1$ admit a cartesian product in $\|\mathcal{C}\|$, which follows from proposition 5.4.3. □

The assumption that $\|\mathcal{C}\|$ is univalent is in general quite reasonable, indeed we have the following:

Proposition 5.4.6. *If $\mathcal{C}$ be a wild category, then $\|\mathcal{C}\|$ is a category. In other words: if $\mathcal{C}$ is univalent, so is $\|\mathcal{C}\|$.*

Proof. We need to show that for every $a, b : \|\operatorname{Ob}\mathcal{C}\|_1$, the canonical map

$$a = b \rightarrow \operatorname{Iso}_{\|\mathcal{C}\|}(a, b)$$

is an equivalence. Since being an equivalence is a proposition, it is in particular a 1-type, so by the elimination principle for $\| - \|_1$, it suffices to show that for every $x, y : \operatorname{Ob}\mathcal{C}$, the map

$$|x|_1 = |y|_1 \rightarrow \operatorname{Iso}\|\mathcal{C}\|(|x|_1, |y|_1) \tag{5.6}$$

is an equivalence. Using truncation elimination and fact 5.4.4, it is possible to directly construct and equivalence

$$\operatorname{Iso}\|\mathcal{C}\|(|x|_1, |y|_1) \simeq \|\operatorname{Iso}\mathcal{C}(x, y)\|_0 \tag{5.7}$$

. Then, we can compose equivalences as follows:

$$
\begin{aligned}
(|x|_1 = |y|_1) &\simeq \|x = y\|_0 && \text{(by fact 4.11.10)} \\
&\simeq \|\operatorname{Iso}\mathcal{C}(x, y)\|_0 && \text{(by univalence of } \mathcal{C}) \\
&\simeq \operatorname{Iso}\|\mathcal{C}\|(|x|_1, |y|_1) && \text{(by (5.7))}
\end{aligned}
$$

Unfolding the definition of the underlying function of this composition of equivalences, one can check that it is indeed equal to the map (5.6), which concludes the proof. □

Remark 5.4.7. While terminal objects and cartesian products in $\mathcal{C}$ are preserved in $\|\mathcal{C}\|$, that is not the case in general for pullbacks. For instance, consider the following diagram of types in the wild category $\mathcal{U}$:

$$
\begin{array}{ccc}
& & 1 \\
& & \downarrow{\scriptstyle \text{base}} \\
1 & \xrightarrow{\text{base}} & \mathsf{S}^1
\end{array}
$$

As noted in example 5.3.17, its pullback is the loop space $\Omega(\mathsf{S}^1, \text{base})$, which is equivalent to the type $\mathbb{Z}$. However, we have

$$\|\operatorname{Hom}_{\mathcal{U}}(\mathsf{S}^1, \mathbb{Z})\|_0 \simeq \mathbb{Z},$$

while

$$\|\operatorname{Hom}_{\mathcal{U}}(\mathsf{S}^1, 1)\|_0 \times_{\|\operatorname{Hom}_{\mathcal{U}}(\mathsf{S}^1, \mathsf{S}^1)\|_0} \|\operatorname{Hom}_{\mathcal{U}}(\mathsf{S}^1, 1)\|_0 \simeq 1 \times_{\mathbb{Z}} 1$$

$$\simeq 1$$

so by the contrapositive of proposition 5.3.21, $|\mathbb{Z}|_1$ is not a pullback of the diagram

$$
\begin{array}{ccc}
& & |1|_1 \\
& & \downarrow \text{base} \\
|1|_1 & \xrightarrow[\text{base}]{} & |S^1|_1
\end{array}
$$

in the category $\|\mathcal{U}\|$.

We now list a few propositions on how to carry functorial and natural structures from wild precategories to their truncations. We do not enter into the detail of the proofs, as they are essentially repeated and tedious applications of the truncation elimination principle (theorem 4.11.3).

Proposition 5.4.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between wild precategories. It induces a functor $\|F\| : \|\mathcal{C}\| \to \|\mathcal{D}\|$.*

Proof. $\|F\|$ is defined on objects and morphisms respectively by using the elimination principles for $\| - \|_1$ and $\| - \|_0$. It is then a straightforward exercise in using the truncation elimination principle to show that $\|F\|$ preserves identities and composition. $\square$

Proposition 5.4.9. *Let $\varphi : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ be a natural transformation between functors between wild precategories. φ induces a natural transformation $\|\varphi\| : \|F\| \Rightarrow \|G\|$.*

Proposition 5.4.10. *Identity functors and the composition of functors are preserved by the mapping $F \mapsto \|F\|$. Identity natural transformations, vertical composition and whiskering are preserved by the mapping $\varphi \mapsto \|\varphi\|$.*

Corollary 5.4.11. *Any monad on a wild precategory $\mathcal{C}$ induces a monad on its truncation $\|\mathcal{C}\|$.*

Proposition 5.4.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be wild precategories. There is a canonical equivalence of precategories*

$$\|\mathcal{C} \times \mathcal{D}\| \simeq \|\mathcal{C}\| \times \|\mathcal{D}\|.$$

Proof. The proof relies once again on repeated uses of the truncation elimination principle, together with fact 5.4.4. $\square$

Corollary 5.4.13. *Any symmetric monoidal structure on a wild precategory $\mathcal{C}$ induces a symmetric monoidal structure on $\|\mathcal{C}\|$.*

Proposition 5.4.14. *If $\mathcal{C}$ is a symmetric monoidal wild precategory, and $x, y : \mathrm{Ob}\,\mathcal{C}$ admit an internal hom $x \multimap y$, then $|x|_1$ and $|y|_1$ admit an internal hom in $\|\mathcal{C}\|$, given by $|x \multimap y|_1$.*

Proof. Using the characterization of internal homs from remark 5.2.6, we need to show that for every $t : \|\mathrm{Ob}\,\mathcal{C}\|_1$, the map

$$\mathrm{Hom}_{\|\mathcal{C}\|}(t, |x \multimap y|_1) \to \mathrm{Hom}_{\|\mathcal{C}\|}(t \|\otimes\| |x|_1, |y|_1)$$

is an equivalence. Using the same reasoning as in the proof of proposition 5.4.3, being an equivalence is a proposition, so by truncation elimination we just have to prove that this holds when t is of the form $|t'|_1$ for every $t' : \mathrm{Ob}\,\mathcal{C}$. Let $t : \mathrm{Ob}\,\mathcal{C}$. Unfolding the definitions of $\mathrm{Hom}_{\|\mathcal{C}\|}$ and $\|\otimes\|$, we are left with showing that the following canonical map is an equivalence:

$$\|\mathrm{Hom}_{\mathcal{C}}(t, x \multimap y)\|_0 \, to \, \|\mathrm{Hom}_{\mathcal{C}}(t \otimes x, y)\|_0.$$

This map is equal to the action of $\|-\|_0$ on the equivalence

$$\mathrm{Hom}_{\mathcal{C}}(t, x \multimap y) \, to \, \mathrm{Hom}_{\mathcal{C}}(t \otimes x, y),$$

so it is itself an equivalence, thus concluding the proof. $\qquad\square$

Corollary 5.4.15. *If $\mathcal{C}$ is a closed symmetric monoidal wild precategory such that $\|\mathcal{C}\|$ is univalent, then $\|\mathcal{C}\|$ is closed symmetric monoidal.*

Proof. The proof is analogous to the proof of proposition 5.4.5 for the case of cartesian products: we need to prove that for every $x, y : \|\mathrm{Ob}\,\mathcal{C}\|_1$, x and y admit an internal hom in $\|\mathcal{C}\|$. Since $\|\mathcal{C}\|$ is univalent, the data of an internal hom is propositional (proposition 5.2.8), so we can use truncation elimination and admit that $x :\equiv |x'|_1, y :\equiv |y'|_1$, and then conclude by proposition 5.4.14. $\qquad\square$

5.5 Wild semantics of linear logic

As we saw in section 3.4, there exist multiple axiomatizations for categorical models of linear logic. Some are easier to adapt to the wild categorical world than others. For instance, Lafont's axiomatization relies on the definition of cofree commutative comonoids, and while we can define wild version of commutative comonoids, the cofree ones will not coincide with actual cofree commutative comonoids from higher category theory. Instead, the axiomatization that seems the simplest to work with in the wild categorical setting is Seely's axiomatization (definition 3.4.3), which we can adapt word-for-word:

Definition 5.5.1. A *Seely wild precategory* consists of:

1. a closed symmetric monoidal wild precategory $(\mathcal{C}, \otimes, 1, \multimap)$,

2. with a terminal object $\top$ and cartesian products $x \,\&\, y$,

3. a comonad $(!, \delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$

4. a strongly symmetric monoidal structure (m^0, m^2) on the functor $! : (\mathcal{C}, \&) \to (\mathcal{C}, \otimes)$,

5. and for every $x, y : \mathrm{Ob}\,\mathcal{C}$, a witness of commutativity for the following diagram:

$$
\begin{array}{ccc}
!x \otimes !y & \xrightarrow{\ \delta_x \otimes \delta_y\ } & !!x \otimes !!y \\
{\scriptstyle m^2_{x,y}}\Big\downarrow & & \Big\downarrow{\scriptstyle m^2_{!x,!y}} \\
!(x \,\&\, y) \xrightarrow[\ \delta_{x\&y}\ ]{} !!(x \,\&\, y) & \xrightarrow[!\langle!\rangle\pi_1, !\pi_2]{} & !(!x \,\&\, !y)
\end{array}
$$

Even though it is not entirely clear in which sense a Seely wild precategory is a "model of linear logic", we can nonetheless recover an actual model of linear logic from a wild one:

Theorem 5.5.2. *Let $\mathcal{C}$ be a Seely wild category (i.e. a Seely wild precategory whose underlying wild precategory is univalent). It homotopy category $\|\mathcal{C}\|$ inherits a canonical Seely category structure.*

Proof. Since $\mathcal{C}$ is univalent, so is $\|\mathcal{C}\|$ by proposition 5.4.6. Hence we can notably apply proposition 5.4.5 and corollary 5.4.15 to recover the cartesian products and closed symmetric monoidal structure on $\|\mathcal{C}\|$. The rest is essentially an amalgamation of all the results of section 5.4. $\qquad\square$

5.6 Wild categories of spans

Now that we have set up the categorical setting in which to define homotopical models of linear logic in homotopy type theory, we are ready to begin the construction of such a model. As explained in section 3.7, it is quite natural to generalize categories of relations to (higher) categories of spans, so in this section we detail how to construct the wild category of spans in a wild category with pullbacks. We will use the same approach we used in section 3.6 for categories of internal relations, and as we will see most proofs will actually look like easier variants of the proofs we carried out in section 3.6, as there is no need to consider regular epimorphisms here.

5.6.1 Definitions

Definition 5.6.1. Given a wild category $\mathcal{C}$ with pullbacks, its wild precategory of *spans*, noted $\mathrm{Span}\,(\mathcal{C})$ or $\mathrm{Span}\,_{\mathcal{C}}$, is the wild precategory whose objects are the objects of $\mathcal{C}$ and whose morphisms are spans

In other words,

$$\mathrm{Span}\,_{\mathcal{C}}(a,b) :\equiv \sum_{x\,\mathrm{Ob}\,\mathcal{C}} \mathcal{C}(x,a) \times \mathcal{C}(x,b).$$

The identities of $\mathrm{Span}\,_{\mathcal{C}}$ are given by

and composition is induced by the pullback:

$$(5.8)$$

Notation 5.6.2. We write $f : a \nrightarrow b$ as a shorthand for $f : \mathrm{Span}\,_{\mathcal{C}}(a,b)$ to avoid confusion with morphisms $g : a \to b$ in $\mathcal{C}$.

As explained throughout section 5.3, in a wild category, univalence entails that objects defined by universal properties are unique (e.g. terminal objects, cartesian products, pullbacks, internal homs). In particular, the composition defined above is well-defined: we can take $x \times_b y$ to be the center of contraction of the proof that the pair (x,y) admits a pullback in $\mathcal{C}$. Univalence also makes this composition associative and unital: for instance we can use the universal property of the pullback to define canonical isomorphisms

$$(x \times_b y) \times_c z \simeq x \times_b (y \times_c z),$$

which translate to actual equalities in $\mathrm{Ob}\,\mathcal{C}$. In a sense, requiring $\mathcal{C}$ to be univalent avoids the usual need in classical category theory to specify a *choice* of pullbacks for every pullback diagram in $\mathcal{C}$, as this choice is already *unique*, in the homotopical sense of being contractible. Unitality, associativity (and eventual higher coherences) of $\mathrm{Span}\,(\mathcal{C})$ also follow from the univalence of $\mathcal{C}$ and the universal properties of iterated pullbacks.

In addition to $\mathrm{Span}\,(\mathcal{C})$ being a well-defined wild precategory, we can actually show it is itself univalent:
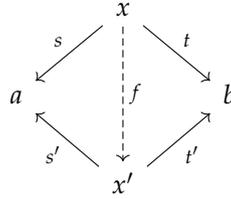
Proposition 5.6.3. *Given a wild category $\mathcal{C}$, the wild precategory $\mathrm{Span}\,(\mathcal{C})$ is univalent.*

To prove this, we will first need a few lemmas:

Lemma 5.6.4. *Let $(x, s, t) : a \nrightarrow b$ and $(x', s', t') : a \nrightarrow b$ be spans from a to b. We have an equivalence*

$$((x, s, t) =_{\mathrm{Span}_{\mathcal{C}}(a,b)} (x', s', t')) \simeq \sum_{f : \mathrm{Iso}_{\mathcal{C}}(x,x')} (s' \circ f = s) \times (t' \circ f = t).$$

In other words, the type of paths from the span (x, s, t) to the span (x', s', t') is given by commutative diagrams of the following shape:

where $f : x \to x'$ is an isomorphism in $\mathcal{C}$.

Proof. Using the characterization of path types in dependent sums (fact 4.5.6), we have

$$(x, s, t) = (x', s', t') \simeq \sum_{p : x = x'} (p_*(s) = s') \times (p_*(t) = t').$$

Since $\mathcal{C}$ is univalent, the data of $p : x = x'$ is equivalent to the data of $f : \mathrm{Iso}_{\mathcal{C}}(x, x')$. Moreover, we can show by path induction that

$$(p_*(s) = s') \simeq (\mathrm{idtoiso}\,(p) \circ s' = s)$$

and similarly for t and t'. Since $\mathrm{idtoiso}\,(\mathrm{ua}\,(f)) = f$, we get a chain of equivalences

$$
\begin{aligned}
((x, s, t) = (x', s', t')) &\simeq \sum_{p : x = x'} (p_*(s) = s') \times (p_*(t) = t') \\
&\simeq \sum_{f : \mathrm{Iso}_{\mathcal{C}}(x,x')} (\mathrm{idtoiso}\,(\mathrm{ua}\,(f)) \circ s' = s) \times (\mathrm{idtoiso}\,(\mathrm{ua}\,(f)) \circ t' = t) \\
&\simeq \sum_{f : \mathrm{Iso}_{\mathcal{C}}(x,x')} (f \circ s' = s) \times (f \circ t' = t),
\end{aligned}
$$

which concludes the proof. □

Lemma 5.6.5. *A span $(x, s, t) : a \nrightarrow b$ is an isomorphism in $\mathrm{Span}\,(\mathcal{C})$ if and only if the morphisms s and t are isomorphisms in $\mathcal{C}$.*

Proof. First suppose that s and t are isomorphisms in $\mathcal{C}$. Then one can straightforwardly check that the following square is a pullback square in $\mathcal{C}$:

$$
\begin{array}{ccc}
a & \xrightarrow{\;s^{-1}\;} & x \\
{\scriptstyle s^{-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\
x & \xrightarrow{\;t\;} & b
\end{array}
$$

so the composition of (x, s, t) and (x, t, s) is given by the diagram

$$
\begin{array}{ccccccc}
 & & & a & & & \\
 & & {\scriptstyle s^{-1}}\swarrow & \vee & \searrow{\scriptstyle s^{-1}} & & \\
 & x & & & & x & \\
 {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & & {\scriptstyle t}\swarrow & & \searrow{\scriptstyle s} \\
 a & & & b & & & a
\end{array}
$$

which by lemma 5.6.4 is indeed equal to the identity span from a to a.

Now, suppose given a span $(x, s, t) : a \twoheadrightarrow b$ with a two-sided inverse $(y, u, v) : b \twoheadrightarrow a$. Using path induction on the proofs of $a = x \times_b y$ and $b = y \times_a x$, we can without loss of generality assume that we have the following composition diagrams of spans:

$$
\begin{array}{ccccccc}
 & & a & & & & \\
 & {\scriptstyle f}\swarrow & \vee & \searrow{\scriptstyle g} & & & \\
 x & & & & y & & \\
 {\scriptstyle s}\swarrow \;\; \searrow{\scriptstyle t} & & & {\scriptstyle u}\swarrow \;\; \searrow{\scriptstyle v} & & & \\
 a \qquad\quad b & & & b \qquad\quad a & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & b & & & & \\
 & {\scriptstyle h}\swarrow & \vee & \searrow{\scriptstyle k} & & & \\
 y & & & & x & & \\
 {\scriptstyle u}\swarrow \;\; \searrow{\scriptstyle v} & & & {\scriptstyle s}\swarrow \;\; \searrow{\scriptstyle t} & & & \\
 b \qquad\quad a & & & a \qquad\quad b & & &
\end{array}
$$

such that the following equalities of morphisms in $\mathcal{C}$ hold:

$$
\begin{aligned}
s \circ f &= \mathrm{id}_a & u \circ h &= \mathrm{id}_b \\
v \circ g &= \mathrm{id}_a & t \circ k &= \mathrm{id}_b
\end{aligned}
$$

Since $s \circ f = \mathrm{id}_a$, f has a left inverse. Now consider the following diagram in $\mathcal{C}$:

$$
\begin{array}{ccccc}
x & \xrightarrow{\;t\;} & b & \xrightarrow{\;h\;} & y \\
{\scriptstyle \mathrm{id}_x}\downarrow & & & & \downarrow{\scriptstyle u} \\
x & & \xrightarrow{\qquad t \qquad} & & b
\end{array}
$$

This diagram commutes because we have

$$
(u \circ h) \circ t = \mathrm{id}_b \circ t = t = t \circ \mathrm{id}_x.
$$

By universal property of the pullback $a = x \times_b y$, there must be a morphism $l : x \to a$ as follows:

In particular, $f \circ l = \mathrm{id}_x$, so f has a right inverse, hence f is an isomorphism in $\mathcal{C}$. Since s is the left inverse of an isomorphism, it must itself be an isomorphism.

A symmetric argument shows that t is also an isomorphism, which concludes the proof. $\qquad\square$

Proof. Proof of proposition 5.6.3 Let a, b be objects of $\mathcal{C}$. We have the following chain of equivalences:

$$
\begin{aligned}
(a \simeq_{\mathrm{Span}_{\mathcal{C}}} b) &\simeq \sum_{x:\mathcal{C}} (x \simeq_{\mathcal{C}} a) \times (x \simeq_{\mathcal{C}} b) && \text{(by lemmas 5.6.4 and 5.6.5)} \\
&\simeq \sum_{x:\mathcal{C}} (x = a) \times (x = b) && \text{(by univalence of } \mathcal{C}) \\
&\simeq \sum_{u:\, \sum_{x:\mathcal{C}} (x=a)} (\pi_1(u) = b) && \text{(reordering terms)} \\
&\simeq (a = b) && \text{(by fact 4.7.5 and corollary 4.8.6)}
\end{aligned}
$$

On can then check that the induced map $a = a \to (a \simeq_{\mathrm{Span}_{\mathcal{C}}} a)$ maps refl_a to the identity span on a, so that the map $a = b \to (a \simeq_{\mathrm{Span}_{\mathcal{C}}} b)$ thus defined is the canonical one from definition 5.1.6. $\qquad\square$

Remark 5.6.6. As evident from the definition, the wild category $\mathrm{Span}(\mathcal{C})$ is self-dual: there is an involutive contravariant functor $\mathrm{Span}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Span}(\mathcal{C})$ acting as the identity on objects and switching the two legs of morphisms.

5.6.2 Functoriality of the Span construction

The goal of this subsection is to address general question of the form:

> "Given some structure on $\mathcal{C}$, does it lift to $\mathrm{Span}(\mathcal{C})$?"

This is the span-theoretical version of what we did in section 3.6.1 for internal relations. Although we will be working with *wild* structures throughout, let us briefly mention as a conjecture what an ∞-categorical version of theorems 3.6.27 and 3.6.29 for higher categories of spans would look like:

Conjecture 5.6.7. *The* Span *construction underlies a limit-preserving $(\infty, 2)$-functor from the $(\infty, 2)$-category of $(\infty, 1)$-categories with pullbacks, functors preserving pullbacks, and cartesian natural transformations, to the $(\infty, 2)$-category of $(\infty, 1)$-categories.*

The results of this section form a wild categorical approximation to this conjecture.

Proposition 5.6.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be wild categories with pullbacks. There is an equivalence of wild categories*

$$\mathrm{Span}(\mathcal{C} \times \mathcal{D}) \simeq \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{D})$$

Proof. $\mathrm{Span}\,(\mathcal{C} \times \mathcal{D})$ and $\mathrm{Span}\,(\mathcal{C}) \times \mathrm{Span}\,(\mathcal{D})$ have the same type of objects. For morphisms, we have a chain of equivalences

$$\mathrm{Span}_{\mathcal{C}}(a,b) \times \mathrm{Span}_{\mathcal{D}}(a',b') \equiv (\sum_{x:\mathrm{Ob}\,\mathcal{C}} (x \to a) \times (x \to b)) \times (\sum_{x':\mathrm{Ob}\,\mathcal{D}} (x' \to a') \times (x' \to b'))$$

$$\simeq (\sum_{(x,x'):\mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}} ((x \to a) \times (x' \to a')) \times ((x \to b) \times (x' \to b))) \quad \text{(reordering terms)}$$

$$\equiv \mathrm{Span}_{\mathcal{C} \times \mathcal{D}}((a,a'),(b,b'))$$

That the induced equivalence respects the composition of spans comes from the fact that pullbacks in $\mathcal{C} \times \mathcal{D}$ are computed pointwise in $\mathcal{C}$ and $\mathcal{D}$. $\qquad\square$

Proposition 5.6.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be wild categories with pullbacks and $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F preserves pullbacks, then it lifts to a functor*

$$\mathrm{Span}\,(F) : \mathrm{Span}\,(\mathcal{C}) \to \mathrm{Span}\,(\mathcal{D}),$$

such that:

- *for $a : \mathrm{Ob}\,\mathcal{C}$,*

$$\mathrm{Span}\,(F)(a) :\equiv F(a)$$

- *for $(x,s,t) : a \twoheadrightarrow b$,*

$$\mathrm{Span}\,(F)(a \xleftarrow{s} x \xrightarrow{t} b) := Fa \xleftarrow{Fs} Fx \xrightarrow{F} Fb$$

Moreover, if $G : \mathcal{D} \to \mathcal{E}$ is another pullback-preserving functor, we have

$$\mathrm{Span}\,(G \circ F) = \mathrm{Span}\,(G) \circ \mathrm{Span}\,(F)$$

and $\mathrm{Span}\,(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\mathrm{Span}\,(\mathcal{C})}$.

Proof. $\mathrm{Span}\,(F)$ preserves identities because F does, and it preserves composition because F preserves pullbacks.

That $\mathrm{Span}\,(-)$ preserves identity functors and the composition of functors is a straightforward computation. $\qquad\square$

To lift natural transformations to wild categories of spans, we first explain how to lift morphisms.

Proposition 5.6.10. *Given objects $a,b : \mathcal{C}$, any morphism $f : a \to b$ canonically induces two spans:*

$$\lambda f : a \twoheadrightarrow b :\equiv a \xleftarrow{\mathrm{id}_a} a \xrightarrow{f} b$$

$$\rho f : b \twoheadrightarrow a :\equiv b \xleftarrow{f} a \xrightarrow{\mathrm{id}_a} a$$

Moreover, we have $\lambda(g \circ f) = \lambda g \circ \lambda f$ since the center square in the following diagram is a pullback square:

This makes λ into a covariant functor $\lambda : \mathcal{C} \to \mathrm{Span}\,(\mathcal{C})$. Similarly, ρ extends to a contravariant functor $\rho : \mathcal{C}^{\mathrm{op}} \to \mathrm{Span}\,(\mathcal{C})$. Actually, λ is just ρ precomposed to the self-duality functor of remark 5.6.6.

To lift natural transformations, we need them to be cartesian.

Definition 5.6.11. A natural transformation $\varphi : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ between functors between wild precategories is said to be *cartesian* if its naturality squares are pullback squares.

Remark 5.6.12. When $\mathcal{D}$ is univalent, being cartesian is a proposition by virtue of proposition 5.3.15.

Proposition 5.6.13. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be pullback-preserving functors between wild categories with pullbacks, and $\varphi : F \Rightarrow G$ be a cartesian natural transformation. Then φ extends to a natural transformation $\mathrm{Span}\,(\alpha) : \mathrm{Span}\,(F) \Rightarrow \mathrm{Span}\,(G)$ with*

$$\mathrm{Span}\,(\varphi)_a :\equiv \lambda\varphi_a : Fa \nrightarrow Ga$$

for every $a : \mathrm{Ob}\,\mathcal{C}$.

Proof. Let $a \xleftarrow{s} x \xrightarrow{t} b$ be a morphism in $\mathrm{Span}\,(\mathcal{C})$. We need to prove that the following square of morphisms in $\mathrm{Span}\,(\mathcal{C})$ commutes:

$$
\begin{array}{ccccc}
Fa & \xleftarrow{\;Fs\;} & Fx & \xrightarrow{\;Ft\;} & Fb \\
{\scriptstyle \mathrm{id}_{Fa}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{id}_{FB}} \\
Fa & & & & Fb \\
{\scriptstyle \varphi_a}\downarrow & & & & \downarrow{\scriptstyle \varphi_b} \\
Ga & \xleftarrow{\;Gs\;} & Gx & \xrightarrow{\;Gt\;} & Gb
\end{array}
\tag{5.9}
$$

We can complete this square in the following way:

where the dotted squares are pullbacks, the middle arrow h is induced by universal property of the pullback $Fa \times_{Ga} Gx$ from the naturality square

$$
\begin{array}{ccc}
Fx & \xrightarrow{\;Fs\;} & Fa \\
{\scriptstyle \varphi_x}\downarrow & & \downarrow{\scriptstyle \varphi_a} \\
Gx & \xrightarrow{\;Gs\;} & Ga
\end{array}
$$

thus making the triangles (1) and (2) commute, and the square (3) commutes by naturality of φ with respect to t. Since φ was assumed to be cartesian, the above naturality square is a pullback square and thus h is an isomorphism. By lemma 5.6.4, the isomorphism h witnesses that the square (5.9) commutes, so that $\mathrm{Span}\,(\varphi)$ is a natural transformation. $\square$

Proposition 5.6.14. *The action on cartesian natural transformations $\varphi \mapsto \mathrm{Span}\,(\varphi)$ respects identities, vertical composition and whiskering up to wild equality.*

Proof. First, identities: let $F : \mathcal{C} \to \mathcal{D}$ be a pullback-preserving functor between wild categories with pullbacks, and let $a : \mathrm{Ob}\,\mathcal{C}$. We have

$$\mathrm{Span}\,(\mathrm{id}_F)_a \equiv \lambda(\mathrm{id}_{F(a)}) \equiv (\mathrm{id}_{\mathrm{Span}\,(F)})_a$$

so identity natural transformations are preserved judgmentally.

Second, for vertical composition of cartesian natural transformations : let $\mathcal{C}, \mathcal{D}$ be wild categories with pullbacks, $F, G, H : \mathcal{C} \to \mathcal{D}$ pullback-preserving functors, $\varphi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ cartesian natural transformations. Given $a : \mathcal{C}$, we have

$$\begin{aligned}
\mathrm{Span}\,(\psi \circ \varphi)_a \equiv \lambda(\psi_a \circ \varphi_a) \qquad &= \lambda(\psi_a) \circ \lambda(\varphi_a)\,(\text{by proposition } 5.6.10) \\
&= \mathrm{Span}\,(\psi_a) \circ \mathrm{Span}\,(\varphi_a),
\end{aligned}$$

so Span preserves vertical composition of cartesian natural transformations up to wild equality.

Finally, for whiskering operations: consider a diagram of wild categories with pullbacks, pullback-preserving functors and natural transformations as follows:

$$\mathcal{B} \xrightarrow{\ K\ } \mathcal{C} \overset{\overset{F}{\longrightarrow}}{\underset{\underset{G}{\longrightarrow}}{\Downarrow \varphi}} \mathcal{D} \xrightarrow{\ H\ } \mathcal{E}$$

- First for left whiskering: since φ is cartesian and H preserves pullbacks, $H * \varphi$ is also cartesian. Moreover, for all $a : \mathrm{Ob}\,\mathcal{C}$, we have

$$\begin{aligned}
\mathrm{Span}\,(H * \varphi)_a &\equiv \lambda(H(\varphi_a)) \\
&= (\mathrm{Span}\,(H))(\lambda(\varphi_a)) \\
&= \mathrm{Span}\,(H)(\mathrm{Span}\,(\varphi)_a) \\
&\equiv (\mathrm{Span}\,(H) * \mathrm{Span}\,(\varphi))_a
\end{aligned}$$

so Span preserves left whiskering up to wild equality.

- Then, for right whiskering: the naturality squares for $\varphi * K$ are naturality squares for φ, so they are pullback squares regardless of the fact that K preserves pullbacks, and $\varphi * K$ is cartesian. Now, for all $a : \mathrm{Ob}\,\mathcal{B}$, we have

$$\begin{aligned}
\mathrm{Span}\,(\varphi * K)_a &\equiv \lambda(\varphi_{Ka}) \\
&= \mathrm{Span}\,(\varphi)_{Ka} \\
&\equiv (\mathrm{Span}\,(\varphi) * \mathrm{Span}\,(K))_a
\end{aligned}$$

so Span also preserves right whiskering up to wild equality.

$\square$

Let $\mathcal{C}$ be a wild category with pullbacks and a terminal object 1. In particular, $\mathcal{C}$ admits finite products, since cartesian products can be computed as pullbacks over 1, and so it admits a cartesian symmetric monoidal structure.

Proposition 5.6.15. *In a wild category $\mathcal{C}$ with a terminal object and pullbacks, the cartesian symmetric monoidal structure lifts to a symmetric monoidal structure on $\mathrm{Span}\,(\mathcal{C})$.*

Proof. The cartesian product functor $- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ commutes with pullbacks by proposition 5.3.23. By proposition 5.6.9, it therefore lifts to a functor

$$\mathrm{Span}\,(- \times -) : \mathrm{Span}\,(\mathcal{C} \times \mathcal{C}) \to \mathrm{Span}\,(\mathcal{C}).$$

Composing $\mathrm{Span}\,(- \times -)$ with the equivalence of proposition 5.6.8, we obtain a tensor product functor

$$- \otimes - : \mathrm{Span}\,(\mathcal{C}) \times \mathrm{Span}\,(\mathcal{C}) \to \mathrm{Span}\,(\mathcal{C}).$$

The unit object of the symmetric monoidal structure is 1, seen as an object of $\mathrm{Span}\,(\mathcal{C})$. The unitality, associativity, braiding, symmetry and all higher coherences for the tensor and unit are lifted from the coherences of the cartesian monoidal structure on $\mathcal{C}$ using proposition 5.6.14. $\qquad\square$

Remark 5.6.16. From the point of view of conjecture 5.6.7, we are using the following idea: a symmetric monoidal ∞-category is a commutative monoid object in the ∞-category of ∞-categories (see the discussion at the beginning of section 6.10 and [Lur17, Remark 2.4.2.6]). The finite products on $\mathcal{C}$ thus induce a structure of commutative monoid object on $\mathcal{C}$ in the ∞-category of ∞-categories with pullbacks and pullback-preserving functors (fact 6.10.15). By conjecture 5.6.7, the ∞-functor Span should be cartesian, and thus preserve commutative monoid objects. Hence it should lift the cartesian symmetric monoidal structure on $\mathcal{C}$ to a symmetric monoidal structure on $\mathrm{Span}\,(\mathcal{C})$. This idea tells us even more: any symmetric monoidal structure on $\mathcal{C}$ that is compatible with pullbacks should lift to a symmetric monoidal structure on $\mathrm{Span}\,(\mathcal{C})$.

Proposition 5.6.17. *The symmetric monoidal wild category* $(\mathrm{Span}\,(\mathcal{C}), \otimes, 1)$ *is closed, with internal hom given by*

$$a \multimap b :\equiv a \otimes b$$

for every $a, b : \mathrm{Ob}\,\mathcal{C}$.

Proof. Given objects a, b, c of $\mathcal{C}$, we have a chain of equivalences

$$
\begin{aligned}
\mathrm{Span}_{\mathcal{C}}(a \otimes b, c) &\equiv \mathrm{Span}_{\mathcal{C}}(A \times B, C) && (\text{definition of } \otimes) \\
&\simeq \sum_{x:\mathcal{C}} (x \to a \times b) \times (x \to c) && (\text{definition of } \mathrm{Span}_{\mathcal{C}}) \\
&\simeq \sum_{x:\mathcal{C}} (x \to a \times b \times c) && (\text{univ. prop. of } \times \text{ in } \mathcal{C}) \\
&\simeq \sum_{x:\mathcal{C}} (x \to a) \times (x \to b \times c) && (\text{univ. prop. of } \times \text{ in } \mathcal{C}) \\
&\simeq \mathrm{Span}_{\mathcal{C}}(a, b \otimes c) && (\text{definition of } \mathrm{Span}_{\mathcal{C}}).
\end{aligned}
$$

Moreover, one can straightforwardly check that the underlying map of this equivalence is equal to the canonical map of remark 5.2.6. $\qquad\square$

Remark 5.6.18. The wild symmetric monoidal category of spans can actually be shown to be *compact closed*, with each object being self-dual, which can be shown following the categorical proof [Sta13]. As such it is closed symmetric monoidal with internal hom being given by

$$
\begin{aligned}
a \multimap b &:\equiv a^* \otimes b \\
&\equiv a \otimes b
\end{aligned}
$$

This also implies than $\mathrm{Span}\,(\mathcal{C})$ is $*$-autonomous. We will not get into more details about compact closedness and $*$-autonomy in the wild categorical setting, as the theory is essentially the same as the classical one. We will however say a bit more about these notions in the ∞-categorical setting in section 6.12.

5.6.3 Finite products in spans of types

We now focus our attention to the case $\mathcal{C} :\equiv \mathcal{U}$ and show that the wild category Span $(\mathcal{U})$ admits finite products.

Proposition 5.6.19. *The wild category* Span $(\mathcal{U})$ *admits finite products, and they are computed as coproducts in* $\mathcal{C}$, *i.e. we have* $\pi_1 = \rho\iota_1 : A \sqcup B \twoheadrightarrow A$ *and* $\pi_2 = \rho\iota_2 : A \sqcup B \twoheadrightarrow B$.

Proof. We first show that Span $(\mathcal{U})$ admits a terminal object given by the empty type $\varnothing$. In a span of types

$$X \xleftarrow{s} Y \xrightarrow{t} \varnothing,$$

the map $t : Y \to \varnothing$ is necessarily unique and implies that $Y \simeq \varnothing$. Hence the data of such a span reduces to that of a map $\varnothing \to X$, and the type $\varnothing \to X$ is contractible by the induction principle for $\varnothing$. Hence $\varnothing$ is a terminal object in Span $(\mathcal{U})$.

We now show that Span $(\mathcal{U})$ admits cartesian products, given by the coproduct of types. Let A and B be types, we have the following chain of equivalences:

$$
\begin{aligned}
\mathrm{Span}_{\mathcal{U}}(X, A \sqcup B) &\simeq (X \times (A \sqcup B)) \to \mathcal{U} && \text{(fibred/indexed equivalence fact 4.9.3)} \\
&\simeq ((X \times A) \sqcup (X \times B)) \to \mathcal{U} \\
&\simeq ((X \times A) \to \mathcal{U}) \times ((X \times B) \to \mathcal{U}) && \text{(by univ. prop. of the coproduct)} \\
&\simeq \mathrm{Span}_{\mathcal{U}}(X, A) \times \mathrm{Span}_{\mathcal{U}}(X, B),
\end{aligned}
$$

which proves that $A \sqcup B$ has the universal property of the cartesian product in Span $(\mathcal{U})$ (proposition 5.3.20). $\qquad\square$

Remark 5.6.20. The proof of proposition 5.6.19 could be more generally performed for any *lextensive* wild category $\mathcal{C}$ (i.e. extensive with a terminal object and pullbacks), but our main focus will be on $\mathcal{C} := \mathcal{U}$ so we do not bother with this level of generality here.

Remark 5.6.21. By self-duality (see remark 5.6.6), proposition 5.6.19 implies that Span $(\mathcal{U})$ also has coproducts, and that they are also given by coproducts of types.

We now have a closed symmetric monoidal wild category Span $(\mathcal{U})$, which is furthermore cartesian. To enhance it to a model of linear logic following definition 5.5.1, we need to equip it with a comonad whose underlying functor is symmetric monoidal from the cartesian structure to the monoidal structure.

5.7 The exponential comonad

On Span $(\mathcal{U})$, the data of a comonad is equivalent to that of a monad, by self-duality (remark 5.6.6). Following the same approach than in section 3.6.2, we begin by defining a monad on $\mathcal{U}$, and then show that it lifts to Span $(\mathcal{U})$ using the results of section 5.6.2.

5.7.1 The exponential endofunctor on $\mathcal{U}$

As we discussed in remarks 3.6.34 and 3.6.42, the multiset functor Mul : Set $\to$ Set does not preserve pullbacks, and as such it cannot be lifted to a functor Span (Set) $\to$ Span (Set). We mentioned in section 3.7 that this could be fixed by working in groupoids instead of sets and defining multisets on X to be pairs

$$(E \in \mathrm{FinSet}, f : E \to X).$$

For this definition to make sense in homotopy type theory, we need a well-behaved notion of finite set.

Definition 5.7.1. Let $n : \mathbb{N}$ be a natural number. We write Fin_n for the *finite set with n elements*, defined inductively by:

$$\mathrm{Fin}_0 \; :\equiv \; \varnothing$$
$$\mathrm{Fin}_{\mathrm{suc}\,(n)} \; :\equiv \; \mathrm{Fin}_n \sqcup 1$$

Definition 5.7.2. Let X be a type. We say that X *is finite* when there is some $n : \mathbb{N}$ such that $\|X \simeq \mathrm{Fin}_n\|_{-1}$, and write

$$\mathrm{isFin}\,(X) :\equiv \sum_{n:\mathbb{N}} \|X \simeq \mathrm{Fin}_n\|_{-1}.$$

Definition 5.7.3. The type of *finite sets* is by definition the type

$$\mathrm{Fin} \; :\equiv \; \sum_{X:\mathcal{U}} \mathrm{isFin}\,(X).$$

Proposition 5.7.4. *For every type X, the type* $\mathrm{isFin}\,(X)$ *is a proposition, so that the projection*

$$\pi_1 : \mathrm{Fin} \to \mathcal{U}$$

is an embedding.

Proof. Let X be a type, and $(m, p), (n, q) : \mathrm{isFin}\,(X)$. $\|X \simeq \mathrm{Fin}_n\|_{-1}$ is a proposition by definition of $\| - \|_{-1}$. Hence by fact 4.7.13 it suffices to show that $m = n$. The type $\mathbb{N}$ is a set by example 4.7.18, so the type $m = n$ is a proposition, hence we can use the elimination principle for truncation on p and q and assume that we have

$$r : X \simeq \mathrm{Fin}_m$$
$$s : X \simeq \mathrm{Fin}_n$$

In particular, we have $\mathrm{Fin}_m \simeq \mathrm{Fin}_n$. An induction on m and n shows that this implies $m = n$. $\qquad\square$

Notation 5.7.5. Because of proposition 5.7.4, when given an element $X : \mathrm{Fin}$, we will also write X for the underlying type $X : \mathcal{U}$, rather than $\pi_1(X) : \mathcal{U}$.

With this, we can define homotopy multisets as follows:

Definition 5.7.6. Let X be a type. The type of *homotopy multisets* on X is the type

$$\mathrm{HMul}\,(X) :\equiv \sum_{E:\mathrm{Fin}} (E \to X).$$

Given a list $(x_1, \dots, x_n)$ of elements of a type X, one can in particular define a map $\mathrm{Fin}_n \to X$ which thus defines an element of $\mathrm{HMul}\,(X)$. This element is written $[x_1, \dots, x_n]$, like for ordinary multisets.

The following proposition makes precise the intuition given in section 3.7.

Proposition 5.7.7. *Let (E, p) and (F, q) be homotopy multisets on a type X. There is an equivalence*

$$(E, p) = (F, q) \; \simeq \; \sum_{f:E\simeq F} q \circ f = p.$$

Proof. By fact 4.5.6, a path from (E, p) and (F, q) corresponds to a path $r : E = F$ together with an equality $r_*(p) = q$. By univalence and path induction, this can shown to be equivalent to the data of an equivalence $f : E \simeq F$ together with an equality $q \circ f = p$. $\qquad\square$

Example 5.7.8. Consider the type $\mathrm{HMul}\,(\mathrm{Bool}\,)$, and write 0 for false and 1 for true for brevity. The element $[0, 0] : \mathrm{Bool}\,$ has two automorphisms, i.e. there are two elements in $[0, 0] = [0, 0]$. Indeed, using the above characterization of paths in $\mathrm{HMul}\,(\mathrm{Bool}\,)$, and using the fact that Bool is a set, the data of such a path is equivalent to the data of an equivalence $\mathrm{Fin}_2 \simeq \mathrm{Fin}_2$, of which there are two (for instance by noticing that $\mathrm{Fin}_2 \simeq \mathrm{Bool}\,$, and referring to example 4.6.3). In particular, the type $\mathrm{HMul}\,(\mathrm{Bool}\,)$ is not a set.

Remark 5.7.9. The type Fin as we defined it is not *small* in the sense that it does not belong to the universe $\mathcal{U}$, but to a larger universe. We could go around this issue by working with a hierarchy of universes, defining

$$\mathrm{Fin} \; :\equiv \; \sum_{X : \mathcal{U}_0} \mathrm{isFin}\,(X) \; : \; \mathcal{U}_1,$$

but we can also directly show that Fin is equivalent to a higher inductive type definable in $\mathcal{U}_0$.

It turns out that working with homotopy multisets defined as families indexed by finite sets is unnecessarily restrictive to model the exponential ! of linear logic: any suitable type of "small types" can work in the place of Fin , and considering such generalizations can give especially interesting examples in homotopy type theory, hence for the remainder of this chapter we will work in this further generality.

Definition 5.7.10. Let $\mathcal{V}$ be a type together with an embedding $\mathrm{El} : \mathcal{V} \hookrightarrow \mathcal{U}$. We will say that a type $X : \mathcal{U}$ is $\mathcal{V}$-*small* if there is some $x : \mathcal{V}$ and an equality $\mathrm{El}\,(x) \simeq X$ (note that "X is $\mathcal{V}$-small" is a proposition since El is an embedding).

Notation 5.7.11. Given an element $X : \mathcal{V}$, we will abuse notation and also write X for the type $\mathrm{El}\,(X) : \mathcal{U}$. Given a type $X : \mathcal{U}$ that belongs to $\mathcal{V}$, we will write $X : \mathcal{V}$, so as to consider $\mathcal{V}$ as if it was a universe of types.

Definition 5.7.12. A *type of small types* or universe of small types is a type $\mathcal{V}$ together with an embedding $\mathrm{El} : \mathcal{V} \hookrightarrow \mathcal{U}$ such that:

1. $\varnothing : \mathcal{V}$

2. $1 : \mathcal{V}$

3. if $X : \mathcal{V}$ and $Y : \mathcal{V}$, $X \sqcup Y : \mathcal{V}$

4. if $X : \mathcal{V}$ and $Y : X \to \mathcal{V}$, $\sum_{x:X} Y(x) : \mathcal{V}$

In other words, $\mathcal{V}$ contains the unit type and is closed under coproducts and dependent sums of types.

Remark 5.7.13. The concept of universe of small types is similar to that of *regular cardinal* in set theory, while the usual universes of type theory are closed to *inaccessible cardinals*.

Example 5.7.14. Fin is a universe of small types (modulo remark 5.7.9).

Axiom 5.7.15. In what follows, we will assume fixed a universe of small types $(\mathcal{V}, \mathrm{El}\,)$.

Notation 5.7.16. Let X be a type. We write $X \xrightarrow{\mathcal{V}} \mathcal{U}$ for the type of type families over X whose total space is $\mathcal{V}$-small, or in other words:

$$X \xrightarrow{\mathcal{V}} \mathcal{U} \; :\equiv \; \sum_{Y:X\to\mathcal{U}} \mathrm{fib}_{\mathrm{El}} \Big(\sum_{x:X} Y(x) \Big).$$

Given an element $Y : X \xrightarrow{\mathcal{V}} \mathcal{U}$, we will abuse notation and also write $Y : X \to \mathcal{U}$ for the underlying type family.

Example 5.7.17. In a type theory with a hierarchy of cumulative universes $(\mathcal{U}_i)_{i\in\mathbb{N}}$, if $\mathcal{U} := \mathcal{U}_j$ for some j, we can take $\mathcal{V} := \mathcal{U}_i$ for any $i < j$ to be a universe of small types in $\mathcal{U}_j$. In that case the map El is just the inclusion of types $\mathcal{U}_i \hookrightarrow \mathcal{U}_j$.

Example 5.7.18. Following the previous example, given universes $\mathcal{U}_0 : \mathcal{U}_1$ and $n \geq -2$, the type

$$\sum_{X:\mathcal{U}_0} \text{is-}n\text{-type}(X)$$

is a universe of small types in $\mathcal{U}_1$.

Example 5.7.19. The type hProp is *not* a universe of small types: even though it contains 1 and is closed under dependent sums, it is clearly not closed under coproducts of types.

We now define the exponential of a type by generalizing definition 5.7.6:

Definition 5.7.20. Given a type A, its *exponential* $!_\mathcal{V} A$ relative to $\mathcal{V}$ is defined to be

$$!_\mathcal{V} A :\equiv \sum_{E:\mathcal{V}} (E \to A)$$

We often simply write $!A$, omitting $\mathcal{V}$ when it is obvious from the context.

Proposition 5.7.21. *The exponential can be extended to an endofunctor* $! : \mathcal{U} \to \mathcal{U}$*, acting on morphisms by postcomposition: given* $f : A \to B$*,*

$$!f : !A \to !B$$
$$(E, p) \mapsto (E, f \circ p)$$

Remark 5.7.22. As already noted, in the case $\mathcal{V} :\equiv \mathrm{Fin}$, we have $!_\mathcal{V} \equiv \mathrm{HMul}$, so $!_\mathcal{V}$ is a generalization of the multiset construction. In section 3.5, we gave two equivalent definitions for the set $\mathrm{Mul}(X)$ of multisets on a set X, each based on one viewpoint from the indexed/fibred correspondence (definitions 3.5.8 and 3.5.9), and the definition of $!_\mathcal{V}$ we gave above generalizes the fibred definition.

The indexed viewpoint defines a multiset on X as a map $m : X \to \mathbb{N}$ with finite support. In homotopy type theory, the notion of finite support does not make as much sense when X is not a set, but using the indexed/fibred equivalence in homotopy type theory (section 4.9), we can build an equivalence:

$$!_\mathcal{V} X \simeq (X \xrightarrow{\mathcal{V}} \mathcal{U}).$$

In the case $\mathcal{V} :\equiv \mathrm{Fin}$, this means that we should consider maps $f : X \to \mathcal{U}$ with *finite total space*, which in the case that X is a set amounts to f having finite support.

Remark 5.7.23. Consider still $\mathcal{V} \equiv \mathrm{Fin}$. Given a set A, the set of multisets on A is the free commutative monoid on A. Similarly, given a type A seen as an ∞-groupoid, we can expect $!A$ to be the free symmetric monoidal ∞-groupoid on A (and similarly we expect $!_\mathcal{V} A$ to be the free symmetric monoid with $\mathcal{V}$-small sums), and we will indeed see that this is the case in remark 6.11.23. Similarly, we can expect $!_\mathcal{V} A$ to be the cofree commutative comonoid on A in $\mathrm{Span}\,(\mathcal{U})$, i.e. that the comonad $!_\mathcal{V}$ is the Lafont exponential. However, as we already noted at the beginning of section 5.5, it is currently unknown how to even state such a result formally in homotopy type theory, since it is unknown how to define homotopy coherent commutative monoids there.

5.7.2 Monad laws

We now explore the monad laws on the endofunctor $!_\mathcal{V} : \mathcal{U} \to \mathcal{U}$. Using the intuition that $!_\mathcal{V}$ generalizes the multiset construction, we define the monad laws by analogy with definition 3.5.13:

$$\eta_A : A \to !A \qquad\qquad \mu_A : !!A \to !A$$
$$a \mapsto (1, \mathrm{cst}_a) \qquad (E, p) \mapsto \left(\left(\sum_{e:E} \pi_1(p(e)) \right), \left((e, e') \mapsto \pi_2(p(e))(e') \right) \right)$$

where $\mathrm{cst}_a : 1 \to A$ denotes the constant map $x \mapsto a$ (which is also equal to $\mathrm{ind}_1(A, a)$). Note that the above definitions rely on the fact that $\mathcal{V}$ contains the unit type and is closed under dependent sums.

Notation 5.7.24. By definition, we have

$$!!A \equiv \sum_{E:\mathcal{V}} \left(E \to \sum_{E':\mathcal{V}} (E' \to A) \right),$$

so an element in $!!A$ consists of a $\mathcal{V}$-small type E, and a family E' of $\mathcal{V}$-small types indexed by E, together with a "coloring map" associating to each pair $(e, e') : \sum_{e:E} E'(e)$ a "color" in A.

In other words, we have an equivalence

$$!!A \simeq \sum_{E:\mathcal{V}} \sum_{E':E\to\mathcal{V}} (\Sigma E E' \to A).$$

Under this equivalence, the multiplication map $\mu : !!A \to !A$ acts as:

$$!!A \to !A$$
$$(E, E', p) \mapsto (\Sigma E E', p)$$

Throughout the rest of this text, we will often abuse notation by writing $(E, E', p) : !!A$, implicitly using this equivalence.

Lemma 5.7.25. *The maps η_A and μ_A above define natural transformations*

$$\eta : \mathrm{id}_\mathcal{U} \Rightarrow !$$
$$\mu : !! \Rightarrow !$$

Proof. Let $f : A \to B$ be a map of types and consider the following squares:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & !A \\
f \downarrow & & \downarrow !f \\
B & \xrightarrow{\eta_B} & !B
\end{array}
\qquad\qquad
\begin{array}{ccc}
!!A & \xrightarrow{\mu_A} & !A \\
!!f \downarrow & & \downarrow !f \\
!!B & \xrightarrow{\mu_B} & !B
\end{array}
$$

Let $a : A$, we have

$$
\begin{aligned}
(!f)(\eta_A(a)) &\equiv (!f)(1, \mathrm{cst}_a) \\
&\equiv (1, f \circ \mathrm{cst}_a) \\
&\equiv (1, \mathrm{cst}_{f(a)}) \\
&\equiv \eta_B(f(a)),
\end{aligned}
$$

so η is natural.

Let $(E, E', p) : !!A$ (under the equivalence of notation 5.7.24), we have

$$
(!f)(\mu_A(E, E', p)) = (!f)(\Sigma E E', p) = (\Sigma E E', f \circ p)
$$

and

$$
\mu_B((!!f)(E, E', p)) = \mu_B(E, E', f \circ p) = (\Sigma E E', f \circ p)
$$

so μ is also natural, which completes the proof. $\qquad\square$

Remark 5.7.26. Note that defining η_A using cst_a instead of $\mathrm{ind}_1(A, a)$ makes the equality

$$
(!f)(\eta_A(a)) = \eta_B(f(a))
$$

hold judgmentally and not just typally.

Proposition 5.7.27. *The tuple $(!, \mu, \eta)$ defines a monad on $\mathcal{U}$.*

Proof. Let $A : \mathcal{U}$. We need to prove that the following square and triangles commute

$$
\begin{array}{ccc}
!!!A & \xrightarrow{!\mu_A} & !!A \\
\mu_{!A} \downarrow & & \downarrow \mu_A \\
!!A & \xrightarrow{\mu_A} & !A
\end{array}
\qquad\qquad
\begin{array}{ccccc}
!A & \xrightarrow{!\eta_A} & !!A & \xleftarrow{\eta_{!A}} & !A \\
& \searrow{\scriptstyle \mathrm{id}_{!A}} & \downarrow{\scriptstyle \mu_A} & \swarrow{\scriptstyle \mathrm{id}_{!A}} & \\
& & !A & &
\end{array}
$$

Let $(E, p) : !A$. We have

$$
\begin{aligned}
\mu_A(\eta_{!A}(E, p)) &\equiv \mu_A(1, \mathrm{cst}_E, (*, e) \mapsto p(e)) \\
&\equiv \left(\sum_{x:1} E , \ (*, e) \mapsto p(e) \right) \\
&= (E, p)
\end{aligned}
$$

so the right triangle commutes, and

$$
\begin{aligned}
\mu_A((!\eta_A)(E, p)) &\equiv \mu_A(E, e \mapsto 1, (e, *) \mapsto p(e)) \\
&\equiv \left(\sum_{e:E} 1 , \ (e, *) \mapsto p(e) \right) \\
&= (E, p)
\end{aligned}
$$

so the left triangle commutes.

Now let $(E, E', E'', p) : !!!A$. Here we are using a characterization of $!!!A$ similar to that of $!!A$ in notation 5.7.24. In other words, we have

$$E : \mathcal{V}$$
$$E' : E \to \mathcal{V}$$
$$E'' : \Sigma E E' \to \mathcal{V}$$
$$p : \Sigma(\Sigma E E')E'' \to A$$

We have

$$\mu_A(!\mu_A(E, E', E'', p)) = \mu_A(E, \Sigma E' E'', p) = (\Sigma E(\Sigma E' E''), p)$$

and

$$\mu_A(\mu_{!A}(E, E', E'', p)) = \mu_A(\Sigma E E', E'', p) = (\Sigma E(\Sigma E' E''), p)$$

hence the square commutes, which completes the proof. $\square$

We finally construct two isomorphisms in $\mathcal{U}$ (i.e. equivalences) that will become the Seely isomorphisms for Span $(\mathcal{U})$ in the next section.

Proposition 5.7.28. *There is an isomorphism $l^0 : 1 \simeq !\varnothing$ defined by*

$$l^0(\star) :\equiv (\varnothing, \mathrm{id}_\varnothing)$$

and a natural isomorphism $l^2_{A,B} : !A \times !B \simeq !(A \sqcup B)$ for all types $A, B : \mathcal{U}$, defined by:

$$l^2_{A,B}((E, p), (F, q)) := (E \sqcup F, p \sqcup q).$$

Proof. For l^0, we have

$$!\varnothing \simeq (\varnothing \xrightarrow{\mathcal{V}} \mathcal{U}) \simeq 1$$

where the first equivalence is from remark 5.7.22 and the second one follows by the induction principle of the empty type and the fact that it is $\mathcal{V}$-small.

For l^2, given types $A, B : \mathcal{U}$, we have

$$!A \times !B \simeq (A \xrightarrow{\mathcal{V}} \mathcal{U}) \times (B \xrightarrow{\mathcal{V}} \mathcal{U}) \simeq (A \sqcup B \xrightarrow{\mathcal{V}} \mathcal{U}) \simeq !(A \sqcup B)$$

using the facts that coproduct types are wild categorical coproducts (example 5.3.9) and that $\mathcal{V}$ is closed under coproduct types. Unfolding this equivalence, we indeed have that

$$l^2_{A,B}((E, p)(F, q)) \equiv (E \sqcup F, p \sqcup q)$$

Given maps $f : A \to C$ and $g : B \to D$, we need to show the following naturality square commutes.

$$
\begin{CD}
!A \times !B @>{l^2_{A,B}}>> !(A \sqcup B) \\
@V{!f \times !g}VV @VV{!(f \sqcup g)}V \\
!C \times !D @>>{l^2_{C,D}}> !(C \sqcup D)
\end{CD}
$$

Let $((E, p), (F, q)) : {!A} \times {!B}$. We have

$$(!(f \sqcup g))(l^2_{A,B}((E, p), (F, q))) \equiv (!(f \sqcup g))(E \sqcup F, p \sqcup q)$$
$$= (E \sqcup F, (f \circ p) \sqcup (g \circ q))$$

and

$$l^2_{C,D}((!f \times !g)((E, p), (F, q))) \equiv l^2_{C,D}((E, f \circ p), (F, g \circ q))$$
$$= (E \sqcup F, (f \circ p) \sqcup (g \circ q))$$

which concludes the proof. $\qquad\square$

5.7.3   Lifting the exponential monad to spans

We now show that the monad ! on $\mathcal{U}$ lifts to a monad on $\mathrm{Span}\,(\mathcal{U})$.

Proposition 5.7.29. *The functor* $! : \mathcal{U} \to \mathcal{U}$ *preserves pullbacks and therefore, by proposition 5.6.9, lifts to a functor*

$$\mathrm{Span}\,(!) : \mathrm{Span}\,(\mathcal{U}) \to \mathrm{Span}\,(\mathcal{U}).$$

Proof. Let $A \xrightarrow{f} C \xleftarrow{g} B$ be a diagram of types. We have the following chain of equivalences:

$$
\begin{aligned}
{!A} \times_{!C} {!B} &\equiv \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{F:\mathcal{V}} \sum_{q:F\to B} ((E, f \circ p) = (F, g \circ q)) && \text{(example 5.3.16)} \\
&\simeq \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{F:\mathcal{V}} \sum_{q:F\to B} \Big(\sum_{l:E=F} (l_*(f \circ p) = g \circ q) \Big) && \text{(by fact 4.5.6)} \\
&\simeq \sum_{E:\mathcal{V}} \sum_{F:\mathcal{V}} \sum_{l:E=F} \sum_{p:E\to A} \sum_{q:F\to B} (l_*(f \circ p) = g \circ q) && \text{(reordering of terms)} \\
&\simeq \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{q:E\to B} (f \circ p = g \circ q) && \text{(contracting } (F, l) \text{ onto } (E, \mathrm{id}_E)) \\
&\simeq \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{q:E\to B} \Big(\prod_{e:E} (f(p(e)) = g(q(e))) \Big) && \text{(function extensionality)} \\
&\simeq \sum_{E:\mathcal{V}} (E \to A \times_C B) && \text{(proposition 5.3.21)} \\
&\equiv {!(A \times_C B)}
\end{aligned}
$$

and one can check by direct computation that the inverse map of the composite equivalence is indeed the canonical map

$$!(A \times_C B) \to {!A} \times_{!C} {!B}$$

defined by the universal property of the pullback. $\qquad\square$

In what follows, to make notations more readable, we will use the same notation for $! : \mathcal{U} \to \mathcal{U}$ and it lift to $\mathrm{Span}\,(\mathcal{U})$, i.e. $! : \mathrm{Span}\,(\mathcal{U}) \to \mathrm{Span}\,(\mathcal{U})$.

Our aim is now to show that this functor inherits monad laws from the ones of the monad constructed on spans in section 5.7.1. Using propositions 5.6.13 and 5.6.14, we only need to show that the natural transformations η and μ are cartesian.

Proposition 5.7.30. *The natural transformation* $\eta : \mathrm{id}_{\mathcal{U}} \Rightarrow !$ *is cartesian.*

Proof. Let $f : A \to B$ be a map between types. Unwinding the corresponding naturality square of η, we get

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
\downarrow{\scriptstyle a \mapsto (1, * \mapsto a)} & & \downarrow{\scriptstyle b \mapsto (1, * \mapsto b)} \\
\sum_{E:\mathcal{V}} (E \to A) & \xrightarrow[(E,p) \mapsto (E, f \circ p)]{} & \sum_{E:\mathcal{V}} (E \to B)
\end{array}
$$

That square is judgmentally commutative, as noted in the proof of lemma 5.7.25. We have the following chain of equivalences:

$$
\begin{aligned}
!A \times_{!B} B &\equiv \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{b:B} \left((E, f \circ p) = (1, \mathrm{cst}_b) \right) && \text{(unfolding the definitions)} \\
&\simeq \sum_{E:\mathcal{V}} \sum_{l:E=1} \sum_{p:E\to A} \sum_{b:B} \left(l_*(f \circ p) = \mathrm{cst}_b \right) && \text{(equality in Σ-types, reordering)} \\
&\simeq \sum_{p:1\to A} \sum_{b:B} \left(f \circ p = \mathrm{cst}_b \right) && \text{(contracting (E,l) onto $(1, \mathrm{id}_1)$)} \\
&\simeq \sum_{a:A} \sum_{b:B} \left(f(a) = b \right) && \text{(induction principle of 1)} \\
&\simeq A && \text{(contracting the last two terms onto $(f(a), \mathrm{refl})$)}
\end{aligned}
$$

Moreover, computing this chain of equivalences from bottom to top exactly gives the connecting map $A \to !A \times_{!B} B$ of the pullback of the naturality square of η, so η is indeed cartesian. $\qquad\square$

Proposition 5.7.31. *The natural transformation $\mu : !! \Rightarrow !$ is cartesian.*

Proof. We use colors to help follow the ordering and reordering the terms in iterated sums:

$$
\begin{aligned}
!A \times_{!B} !!B &\simeq \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{F:\mathcal{V}} \sum_{F':F\to\mathcal{V}} \sum_{q:\Sigma FF'\to B} (E, f \circ p) =_{!B} (\Sigma FF', q) && \text{(see notation 5.7.24 for $!!B$)} \\
&\simeq \sum_{E:\mathcal{V}} \sum_{p:E\to A} \sum_{F:\mathcal{V}} \sum_{F':F\to\mathcal{V}} \sum_{q:\Sigma FF'\to B} \sum_{l:E=\Sigma FF'} l_*(f \circ p) = q && \text{(equality in Σ-types)} \\
&\simeq \sum_{F:\mathcal{V}} \sum_{F':F\to\mathcal{V}} \sum_{p:\Sigma FF'\to A} \sum_{q:\Sigma FF'\to B} f \circ p = q && \text{(contracting (E, l) onto $(\Sigma FF', \mathrm{id})$)} \\
&\simeq \sum_{F:\mathcal{V}} \sum_{F':F\to\mathcal{V}} \left(\Sigma FF' \to A \right) && \text{(contracting $(q, -)$ onto $(f \circ p, \mathrm{refl})$)} \\
&\simeq !!A && \text{(notation 5.7.24)}
\end{aligned}
$$

Like with η, one can check that the underlying map of the inverse of the equivalence thus constructed is equal to the map $!!A \to !A \times_{!B} !!B$ induced by the naturality square of μ. Hence μ is cartesian. $\qquad\square$

Together with propositions 5.6.13 and 5.6.14, we thus get:

Corollary 5.7.32. *The natural transformations η and μ lift to $\mathrm{Span}(\mathcal{U})$, and the tuple $(!, \mathrm{Span}(\eta), \mathrm{Span}(\mu))$ defines a monad on $\mathrm{Span}(\mathcal{U})$.*

By remark 5.6.6, ! also defines a comonad on $\mathrm{Span}\,(\mathcal{U})$. We write ε and δ respectively for the counit and comultiplication of the comonad !. Unfolding the definitions, we have

$$\varepsilon_A : !A \twoheadrightarrow A \qquad\qquad\qquad \delta_A : !A \twoheadrightarrow !!A$$

$$\varepsilon_A :\equiv \rho(\eta_A) = !A \xleftarrow{\eta_A} A \xrightarrow{\mathrm{id}_A} A \qquad\qquad \delta_A :\equiv \rho(\mu_A) = !A \xleftarrow{\mu_A} !!A \xrightarrow{\mathrm{id}_{!!A}} !!A$$

Similarly, l^2 being a natural isomorphism, its naturality squares are pullback squares so that, under the equivalence

$$\mathrm{Span}\,(\mathcal{U} \times \mathcal{U}) \simeq \mathrm{Span}\,(\mathcal{U}) \times \mathrm{Span}\,(\mathcal{U})$$

of proposition 5.6.8 and by self-duality (remark 5.6.6), it lifts to a natural isomorphism

$$m^2_{A,B} := \rho(l^2_{A,B}) : !(A \sqcup B) \twoheadrightarrow !A \times !B.$$

The isomorphism l^0 also lifts to an isomorphism

$$m^0 := \rho(l^0) : !\varnothing \twoheadrightarrow 1.$$

Theorem 5.7.33. *The closed symmetric monoidal, cartesian wild category*

$$(\mathrm{Span}\,(\mathcal{U}), \otimes, 1, \sqcup, \varnothing)$$

equipped with the comonad $(!, \delta, \varepsilon)$ *and the isomorphisms* m^2, m^0 *is a Seely wild category.*

Proof. We need to check that

$$(!, m^2, m^0) : (\mathrm{Span}\,(\mathcal{U}), \sqcup, \varnothing) \to (\mathrm{Span}\,(\mathcal{U}), \otimes, \mathbf{1})$$

is strongly symmetric monoidal, and that for all $A, B : \mathcal{U}$ the following diagram commutes:

$$
\begin{array}{ccccc}
!(A \sqcup B) & \xrightarrow{\ \delta_{A \sqcup B}\ } & !!(A \sqcup B) & \xrightarrow{\ !(!\pi_1, !\pi_2)\ } & !(!A \sqcup !B) \\
{\scriptstyle m^2_{A,B}}\Big\downarrow & & & & \Big\downarrow{\scriptstyle m^2_{!A,!B}} \\
!A \times !B & & \xrightarrow{\qquad\ \delta_A \times \delta_B\ \qquad} & & !!A \times !!B
\end{array}
$$

But this diagram is precisely the image by the functor $\rho : \mathcal{U}^{\mathrm{op}} \to \mathrm{Span}\,(\mathcal{U})$ of the diagram

$$
\begin{array}{ccccc}
!(A \sqcup B) & \xleftarrow{\ \mu_{A \sqcup B}\ } & !!(A \sqcup B) & \xleftarrow{\ !\langle!\iota_1, !\iota_2\rangle\ } & !(!A \sqcup !B) \\
{\scriptstyle l^2_{A,B}}\Big\uparrow & & & & \Big\uparrow{\scriptstyle l^2_{!A,!B}} \\
!A \times !B & & \xleftarrow{\qquad\ \mu_A \times \mu_B\ \qquad} & & !!A \times !!B
\end{array}
$$

Let $\mathbb{E} := (E, E', p) : !!A$ and $\mathbb{F} :\equiv (F, F', q) : !!B$ (under the equivalence of notation 5.7.24). We have

$$l^2_{A,B}((\mu_A \times \mu_B)(\mathbb{E}, \mathbb{F})) = l^2_{A,B}((\Sigma E E', p), (\Sigma F F', q)) = (\Sigma E E' \sqcup \Sigma F F', p \sqcup q)$$

and

$$!!A \times !!B \xrightarrow{\ l^2_{!A,!B}\ } !(!A \sqcup !B) \xrightarrow{\ !\langle!\iota_1, !\iota_2\rangle\ } !!(A \sqcup B) \xrightarrow{\ \mu_{A \sqcup B}\ } !(A \sqcup B)$$

$$(\mathbb{E}, \mathbb{F}) \longmapsto (E \sqcup F, (E', p) \sqcup (F', q)) \longmapsto (E \sqcup F, \langle E', F'\rangle, p \sqcup q) \longmapsto \big(\textstyle\sum_{x : E \sqcup F}\ \sum_{y : \langle E', F'\rangle(x)}\ (p \sqcup q)(x, y)\big)$$

Those two elements of $!(A \sqcup B)$ are equal by virtue of dependent sums distributing over disjoint sums.

What remains to be shown is that the following data constitutes a strongly symmetric monoidal functor:

$$(!, m^2, m^0) : (\mathrm{Span}\,(\mathcal{U}), \sqcup, \varnothing) \to (\mathrm{Span}\,(\mathcal{U}), \times, \mathbf{1}).$$

Since the functor ! on spans is obtained as a lifting of $! : \mathcal{U} \to \mathcal{U}$, and similarly for the natural transformations m^2 and m^0, using proposition 5.6.14 we only need to show that

$$(!, l^2, l^0) : (\mathcal{U}, \sqcup, \varnothing) \to (\mathcal{U}, \times, \mathbf{1})$$

is strongly symmetric monoidal. In other words, writing $\alpha, \lambda, \rho, \gamma$ respectively for the associator, left and right unitor, and braiding of the $\sqcup$ and $\times$ symmetric monoidal structures on $\mathcal{U}$, we need show that the following four diagrams commute:

$$
\begin{array}{ccc}
(!A \times !B) \times !C & \xrightarrow{\ \alpha_\times\ } & !A \times (!B \times !C) \\
{\scriptstyle l^2_{A,B} \times \mathrm{id}_{!C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{!A} \times l^2_{B,C}} \\
!(A \sqcup B) \times !C & \quad (1) \quad & !A \times !(B \sqcup C) \\
{\scriptstyle l^2_{A \sqcup B, C}} \downarrow & & \downarrow {\scriptstyle l^2_{A, B \sqcup C}} \\
!((A \sqcup B) \sqcup C) & \xrightarrow[\ !\alpha_\sqcup\ ]{} & !(A \sqcup (B \sqcup C))
\end{array}
$$

$$
\begin{array}{ccc}
!A \times \mathbf{1} & \xrightarrow{\ \rho_\times\ } & !A \\
{\scriptstyle \mathrm{id}_{!A} \times l^0} \downarrow & (2) & \uparrow {\scriptstyle !\rho_\sqcup} \\
!A \times !\varnothing & \xrightarrow[\ l^2_{A, \varnothing}\ ]{} & !(A \sqcup \varnothing)
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{1} \times !B & \xrightarrow{\ \lambda_\times\ } & !B \\
{\scriptstyle l^0 \times \mathrm{id}_{!B}} \downarrow & (3) & \uparrow {\scriptstyle !\lambda_\sqcup} \\
!\varnothing \times !B & \longrightarrow & !(\varnothing \sqcup B)
\end{array}
$$

$$
\begin{array}{ccc}
!A \times !B & \xrightarrow{\ \gamma_\times\ } & !B \times !A \\
{\scriptstyle l^2_{A,B}} \downarrow & (4) & \downarrow {\scriptstyle l^2_{B,A}} \\
!(A \sqcup B) & \xrightarrow[\ !\gamma_\sqcup\ ]{} & !(B \sqcup A)
\end{array}
$$

Most verifications are straightforward:

- for diagram (1), given $(((E, p), (F, q)), (G, r)) : (!A \times !B) \times !C$, both paths in the hexagon evaluate to $(E \sqcup (F \sqcup G), p \sqcup (q \sqcup r)) : !(A \sqcup (B \sqcup C))$ up to associativity of $\sqcup$, which is an equivalence and hence an equality by univalence (see the case of (4) for more a detailed reasoning along those lines),

- diagrams (2) and (3) commute because $E \sqcup \varnothing \simeq E \simeq \varnothing \sqcup E$ in $\mathcal{V}$,

- and finally for diagram (4), given $((E, p), (F, q)) : !A \times !B$, we have

$$l^2_{B,A}(\gamma_\times((E, p), (F, q))) = l^2_{B,A}((F, q), (E, p)) = (F \sqcup E, q \sqcup p)$$

and

$$!\gamma_\sqcup(l^2_{A,B}((E, p), (F, q))) = !\gamma_\sqcup(E \sqcup F, p \sqcup q) = (E \sqcup F, \gamma_\sqcup \circ (p \sqcup q))$$

By characterization of equality in dependent sums (fact 4.5.6) and univalence, an equality

$$(E \sqcup F, \gamma_\sqcup \circ (p \sqcup q)) \ = \ (F \sqcup E, q \sqcup p)$$

in $!(B \sqcup A)$ consists of the data of:

- an equivalence $f : E \sqcup F \simeq F \sqcup E$,

– an equality $e : \gamma_\sqcup \circ (p \sqcup q) = (q \sqcup p) \circ f$ in $E \sqcup F \to B \sqcup A$.

Choosing $f :\equiv \gamma_\sqcup$, a direct computation shows that the following square commutes

$$
\begin{array}{ccc}
E \sqcup F & \xrightarrow{\ p \sqcup q\ } & A \sqcup B \\
{\scriptstyle \gamma_\sqcup}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_\sqcup} \\
F \sqcup E & \xrightarrow[\ q \sqcup p\ ]{} & B \sqcup A
\end{array}
$$

which concludes the proof.

$\square$

Theorem 5.7.33 is the statement that $\mathrm{Span}\,(\mathcal{U})$ forms a "wild model" of intuitionistic linear logic. In addition it is also $*$-autonomous (remark 5.6.18), and it admits products (proposition 5.6.19) and coproducts (remark 5.6.21), so it is full *wild* model of classical linear logic. To get a an actual categorical model in the classical 1-categorical sense, it suffices to apply proposition 5.4.6 and theorem 5.5.2:

Theorem 5.7.34. *The category $\|\mathrm{Span}\,(\mathcal{U})\|_1$ is a $*$-autonomous Seely category with products and coproducts, and thus a full model of classical linear logic.*

This model is of a very different nature from other 1-categorical ones: the category $\|\mathcal{U}\|$ embeds faithfully in $\|\mathrm{Span}\,(\mathcal{U})\|$ through the functor $\|\rho\|$, and the category $\|\mathcal{U}\|$ is known to not be *concrete*: it does not admit a faithful functor to the category of sets. Indeed when interpreting homotopy type theory in set theory using for instance the model of simplicial sets ([KL21]), $\|\mathcal{U}\|$ corresponds to the homotopy category of spaces, which has been shown to not be concrete in [Fre70]. This implies that the category $\|\mathrm{Span}\,(\mathcal{U})\|$ is itself not concrete. In other words, the objects of our model cannot be described as sets with additional structures.

Remark 5.7.35. In the case $\mathcal{V} :\equiv \mathrm{Fin}$, and restricting $\mathcal{U}$ to only consist of 1-types, the Seely wild category $\mathrm{Span}\,(\mathcal{U})$ is actually a $(3,1)$-category that coincides with a special case of Melliès's template games model described in [Mel19]. Our work generalizes this both at the level of objects (where we consider higher homotopy types), and at the level of the exponential, where $!_\mathcal{V}$ describes the act of duplicating resources in a way that can itself be inherently homotopical.

5.8 The exponential modality and polynomials

In this section, we are interested in explicitly describing the non-linear morphisms in the Seely wild category of theorem 5.7.33. More specifically, we show that the wild Kleisli category associated to the comonad $!_\mathcal{V}$ on $\mathrm{Span}\,(\mathcal{U})$ coincides with the wild category of well-known *polynomials in types*.

Definition 5.8.1. Let I, J be types. A *polynomial* from I to J is a diagram of the shape in $\mathcal{U}$:

$$
\begin{array}{ccccc}
 & & E & \xrightarrow{\ p\ } & B \\
 & {\scriptstyle s}\swarrow & & & \searrow{\scriptstyle t} \\
I & & & & J
\end{array}
\tag{5.10}
$$

In other words, the type of polynomials with source I and target J is

$$
\mathrm{Poly}\,(I, J) :\equiv \sum_{E:\mathcal{U}} \sum_{B:\mathcal{U}} (E \to I) \times (E \to B) \times (B \to J)
$$

To make sense of this definition, one must understand what the data of a polynomial represents, namely, a *polynomial functor*.

5.8.1   The functor associated to a polynomial

In what follows, given a map $f : A \to B$ and $b : B$, we write A_b for $\mathrm{fib}_f(b)$, the function f being often implicit from the context.

Definition 5.8.2. Any polynomial $P = (E, B, s, t, p)$ as in eq. (5.10) induces a map

$$(\! | P | \!) : \mathcal{U}^I \to \mathcal{U}^J,$$

defined by

$$(\! | P | \!)(X)(j) :\equiv \sum_{b:B_j} \prod_{e \in E_b} X(s(e))$$

and called the *polynomial functor* induced by P.

Remark 5.8.3. Technically we only defined $(\! | P | \!)$ to be a map and not a functor, but we can define a canonical wild category structure on $\mathcal{U}^I$, and $(\! | P | \!)$ can be enhanced to an actual functor between wild categories.

Remark 5.8.4. This definition of polynomials and polynomial functors is due to Nicola Gambino and Joachim Kock in their article "Polynomial functors and polynomial monads" [GK13]. There, they work in an arbitrary locally cartesian closed category $\mathcal{C}$, and every polynomial P in $\mathcal{C}$ induces a polynomial functor between slice categories

$$(\! | P | \!) : \mathcal{C}_{/I} \to \mathcal{C}_{/J}.$$

In the case of Set, the indexed/fibred correspondence gives a natural equivalence of categories $\mathrm{Set}^I \simeq \mathrm{Set}_{/I}$, so we can also write $(\! | P | \!) : \mathrm{Set}^I \to \mathrm{Set}^J$. Similarly, the indexed/fibred correspondence in homotopy type theory (fact 4.9.3) gives an equivalence

$$\mathcal{U}^I \simeq \sum_{X:\mathcal{U}} (X \to I),$$

and definition 5.8.2 is the indexed variant of the definition of [GK13].

Example 5.8.5. Let $I \equiv J \equiv 1$, and let P denote the following diagram of sets in $\mathcal{U}$

$$1 \longleftarrow \mathrm{Fin}_3 \longrightarrow \mathrm{Fin}_3 \longrightarrow 1$$

Then for every $X : \mathcal{U}$, we have

$$(\! | P | \!)(X) \simeq X \times X \sqcup X \sqcup 1.$$

Example 5.8.6. Now let $I \equiv J \equiv \text{Bool}$, and P be the following diagram of sets in $\mathcal{U}$:

$$\text{Bool} \longleftarrow \text{Fin}_4 \longrightarrow \text{Fin}_3 \longrightarrow \text{Bool}$$
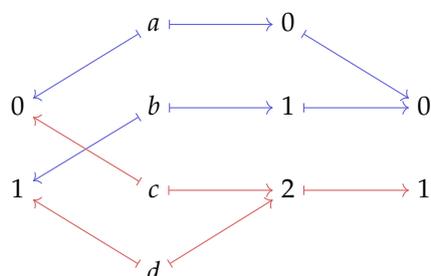
Then, under the identification $\mathcal{U}^{\text{Bool}} \simeq \mathcal{U} \times \mathcal{U}$, we have that

$$\langle\!\langle P \rangle\!\rangle(X, Y) = (X \times Y, X \sqcup Y).$$

From these two examples, we see that the elements of B represent the *monomials* of the polynomial P, the elements of E represent the *arities* of these monomials, and the types I and J represent its *input variables* and *output "colors"*.

Polynomials can be useful to represent data structures in the context of programming languages, where they are also known as *containers*.

Example 5.8.7. Let L denote the following polynomial:

$$1 \longleftarrow \sum_{n:\mathbb{N}} \text{Fin}_n \xrightarrow{\;\pi_1\;} \mathbb{N} \longrightarrow 1$$

We have

$$\langle\!\langle L \rangle\!\rangle(X) :\equiv \sum_{n:\mathbb{N}} X^n,$$

which can be recognized as the type of *lists* on X.

Working in homotopy type theory, we can find interesting examples of polynomial functors based on higher types.

Example 5.8.8. The polynomial functor associated to the polynomial H:

$$1 \longleftarrow \sum_{E:\text{Fin}} E \xrightarrow{\;\pi_1\;} \text{Fin} \longrightarrow 1$$

is by definition

$$\langle\!\langle H \rangle\!\rangle(X) :\equiv \sum_{E:\text{Fin}} (E \to X),$$

i.e. the type $\text{HMul}(X)$ of homotopy multisets on X.

In a sense, fact 4.5.6 allows us to see dependent sums over a 1-type X as *homotopy quotients* by the group actions of $\pi_1(X, x)$. This can be made formal in ∞-category theory, where it can be shown that the dependent sum $\sum_{x:X} Y(x)$ is the ∞-categorical colimit of the diagram of ∞-groupoids $Y : X \to \mathcal{U}$.

Back to linear logic, there is a canonical notion of linearity for polynomials: a polynomial functor $(\!|P|\!)$ can thought of as *linear* when the dependent products in the expression

$$\sum_{\Sigma_{b:B_j} \ \prod_{e:E_b} X(s(e))}$$

are dependent products over contractible types, i.e. when the E_b are contractible. By fact 4.9.5, this is equivalent to asking for the middle map $p : E \to B$ to be an equivalence.

Definition 5.8.9. A polynomial $P = (E, B, s, p, t)$ is said to be *linear* if the map p is an equivalence of types.

Remark 5.8.10. Any span

$$I \xleftarrow{\ s\ } X \xrightarrow{\ t\ } J$$

induces a linear polynomial

$$I \xleftarrow{\ s\ } X \xrightarrow{\ \mathrm{id}_X\ } X \xrightarrow{\ t\ } J$$

Moreover, by univalence one can show that every linear polynomial is equal to one of this form: the canonical map

$$\mathrm{Span}_{\mathcal{U}}(I, J) \to \mathrm{LinPoly}(I, J)$$

is an equivalence.

We can generalize definition 5.8.9 by weakening the requirements on the fibers of p, i.e. by selecting which *arities* are allowed for the products appearing the associated polynomial functor.

Definition 5.8.11. Let $\mathcal{V} \hookrightarrow \mathcal{U}$ be a type with an embedding in $\mathcal{U}$. A polynomial (E, B, s, p, t) is said to be $\mathcal{V}$-*ary* if the fibers of the map p are $\mathcal{V}$-small. We write $\mathrm{Poly}_{\mathcal{V}}(I, J)$ for the type of $\mathcal{V}$-ary polynomials between I and J. In other words, writing $E \to_{\mathcal{V}} B$ for the type of maps $f : E \to B$ whose fibers are $\mathcal{V}$-small, we have:

$$\mathrm{Poly}_{\mathcal{V}}(I, J) :\equiv \sum_{E:\mathcal{U}} \sum_{B:\mathcal{U}} (E \to I) \times (E \to_{\mathcal{V}} B) \times (B \to J).$$

Example 5.8.12. Here are some examples of universes of arities and the corresponding notions of polynomials:

- if $\mathcal{V}$ is the universe of contractible types, the $\mathcal{V}$-ary polynomials are the linear ones, hence the spans by remark 5.8.10,

- if $\mathcal{V} :\equiv \mathrm{hProp}$ is the universe of propositions, the $\mathcal{V}$-ary polynomials could be called *affine polynomials*,

- if $\mathcal{V} \equiv \mathrm{Fin}$ is the universe of finite sets, then we talk about *finitary polynomial functors*.

Note that we do not need $\mathcal{V}$ to be a universe of small types in the sense of definition 5.7.12 for the notion of $\mathcal{V}$-ary polynomial to make sense.

Remark 5.8.13. The finitary polynomial functors in homotopy type theory were shown in [Fin+21] to form a cartesian closed wild category. This was the starting motivation for this very thesis: since spans form a linear counterpart to polynomials, there remained the question of whether this cartesian closed wild category could fit into a model of linear logic, and this is precisely the content of theorem 5.8.17 below.

Axiom 5.8.14. We now assume fixed a universe of small types $\mathcal{V}$ in the sense of definition 5.7.12, as in section 5.7.

Proposition 5.8.15. *There is a wild category* $\mathrm{Poly}_{\mathcal{V}}$ *whose*

- *objects are types in* $\mathcal{U}$,

- *morphisms are the* $\mathcal{V}$-*ary polynomials,*

- *identities are given by identity spans seen as polynomials:*

$$\mathrm{Id}_A :\equiv A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A$$

- *the composition of two polynomials*

$$P :\equiv I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J \text{ and}$$
$$Q :\equiv J \xleftarrow{u} F \xrightarrow{q} C \xrightarrow{v} K$$

is given by

$$I \xleftarrow{s \circ \pi_2 \circ \pi_2} \sum_{(c,\gamma):D} \sum_{x:F_c} E_{\gamma(x)} \xrightarrow{\pi_1} D \xrightarrow{v \circ \pi_1} K$$

where $D :\equiv \sum_{c:C} \prod_{x:F_c} B_{u(x)}$.

Proof. The unitality and associativity of composition will follow from theorem 5.8.17 (a direct proof is also available in Cubical Agda in our github repository `https://github.com/elies-h/ll-polynomials`).

The fact that $\mathrm{Poly}_{\mathcal{V}}$ is univalent was claimed in [Fin+21] without proof, so we give a proof here. Suppose that

$$P :\equiv I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J \text{ and}$$
$$Q :\equiv J \xleftarrow{u} F \xrightarrow{q} C \xrightarrow{v} I$$

are mutual inverses in $\mathrm{Poly}_{\mathcal{V}}$. Using a characterization of equality of polynomials similar to lemma 5.6.4 for spans, this means that we have commutative diagrams:

where $D :\equiv \sum_{c:C} \prod_{x:F_c} B_{u(x)}$ and $D' :\equiv \sum_{b:B} \prod_{x:E_b} C_{s(x)}$.

Since the map $\pi_1 : \sum_{(b,\beta):D'} \sum_{x:E_b} F_{\beta(x)} \to D'$ is an equivalence, by fact 4.9.5, for every $(b, \beta) : D'$, we have that $\sum_{x:E_b} F_{\beta(x)}$ is contractible, and so in particular E_b is contractible. Now let $b : B$, and let's build some $\beta : \prod_{x:E_b} C_{s(x)}$.

Let $x : E_b$. By commutativity of the triangle (1), the map

$$D \to I$$
$$(c, \gamma) \mapsto v(c)$$

must be an equivalence, so let θ denote an inverse to it. We have $\theta(s(x)) : \sum_{c:C} \prod_{y:F_c} B_{u(x)}$, which we may assume to be of the form (c, γ). Since θ is an inverse to $v \circ fst$, we have an equality $\xi : v(c) = s(x)$, and so we have $(c, \xi) : C_{s(x)}$.

The association $x \mapsto (c, \xi)$ defines a map $\beta : \prod_{x:E_b} C_{s(x)}$, so for every $b : B$ there is some $\beta : \prod_{x:E_b} C_{s(x)}$, and by the reasoning above E_b must be contractible. Since $E_b :\equiv \text{fib}_p(b)$ is contractible for every $b : B$, by fact 4.9.5 this means that $p : E \to B$ is an equivalence. A symmetric argument shows that $q : F \to C$ must also be an equivalence. In other words, P and Q must be linear polynomials. Since linear polynomials compose like spans, the univalence of $\text{Poly}_\mathcal{V}$ follows from that of $\text{Span}(\mathcal{V})$ (proposition 5.6.3). $\qquad\square$

5.8.2 Polynomials are Kleisli morphisms

Proposition 5.8.16. *Let I, J be types. There is an equivalence*

$$\text{poly-to-span} : \text{Poly}_\mathcal{V}(I, J) \simeq \text{Span}(!_\mathcal{V} I, J)$$

which maps a polynomial

$$I \xleftarrow{s} E \xrightarrow{p}_\mathcal{V} B \xrightarrow{t} J$$

to

$$!_\mathcal{V} I \xleftarrow{\bar{s}} B \xrightarrow{t} J$$

with $\bar{s}(b) :\equiv \big(E_b, s|_{E_b}\big)$, where $s|_{E_b}$ is the restriction of s to E_b.

Proof. Fixing $B : \mathcal{U}$, there is an equivalence.

$$\sum_{E:\mathcal{U}} (E \to I) \times (E \to_\mathcal{V} B) \simeq \sum_{F:B \to \mathcal{V}} (\Sigma B F \to I)$$
$$\simeq B \to \sum_{F:\mathcal{V}} (F \to I) \qquad\qquad (5.11)$$
$$\equiv (B \to !_\mathcal{V} I)$$

Thus, we have the following chain of equivalences:

$$\text{Poly}_\mathcal{V}(I, J) \equiv \sum_{E:\mathcal{U}} \sum_{B:\mathcal{U}} (E \to I) \times (E \to_\mathcal{V} B) \times (B \to J)$$
$$\simeq \sum_{B:\mathcal{U}} \Big(\sum_{E:\mathcal{U}} (E \to I) \times (E \to_\mathcal{V} B) \Big) \times (B \to J) \qquad \text{(reordering of terms)}$$
$$\simeq \sum_{B:\mathcal{U}} (B \to !_\mathcal{V} I) \times (B \to J) \qquad\qquad \text{(by eq. (5.11))}$$
$$\equiv \text{Span}(!_\mathcal{V} I, J)$$

which concludes the proof. $\qquad\square$

Now, the interesting point is that this equivalence respects composition and identities:

Theorem 5.8.17. *The Kleisli wild category* $\mathrm{Span}\,(\mathcal{U})_{!_\gamma}$ *associated to the comonad* $!_\gamma$ *on* $\mathrm{Span}\,(\mathcal{U})$ *is equivalent to the wild category* $\mathrm{Poly}\,_\gamma$ *of polynomials.*

Here, the *Kleisli wild category* denotes the wild precategory with the same objects as $\mathrm{Span}\,(\mathcal{U})$, morphisms given by $\mathrm{Span}_\mathcal{U}(!_\gamma I, J)$, and where the identities, composition, unitality and associativity come from the comonad laws on $!_\gamma$. Note that it is not immediately obvious that $\mathrm{Span}\,(\mathcal{U})_{!_\gamma}$ is univalent, and that is more of a consequence of the theorem.

Proof. The wild precategories $\mathrm{Span}\,(\mathcal{U})_{!_\gamma}$ and $\mathrm{Poly}\,_\gamma$ have the same type of objects, namely $\mathcal{U}$, so we can take the identity as the mapping on objects $\mathrm{Poly}\,_\gamma \to \mathrm{Span}\,(\mathcal{U})_{!_\gamma}$. Since it is an equivalence on objects, the functor we are constructing is in particular essentially surjective. The action on morphisms is given by the equivalence of proposition 5.8.16 (thus our functor will be fully faithful). It remains to be shown that this mapping is compatible with identities and composition.

Identities. Fix a type A. By proposition 5.8.16, the identity polynomial Id_A on A is mapped to the span

$$!_\gamma A \xleftarrow{\overline{\mathrm{id}_A}} A \xrightarrow{\mathrm{id}_A} A$$

with $\overline{\mathrm{id}_A}(a) \equiv \big(\sum_{a':A} (a' = a), (a', p) \mapsto a' \big)$. But the type $\sum_{a':A} (a' = a)$ is contractible with center $(a, \mathrm{refl}\,_a)$ by fact 4.7.5, so

$$\overline{\mathrm{id}_A}(a) = (\mathbf{1}, \mathrm{cst}\,_a) = \eta_A(a).$$

So in the end poly-to-span $(\mathrm{Id}_A) = \varepsilon_A$, which is the identity of A in $\mathrm{Span}\,(\mathcal{U})_{!_\gamma}$.

Composition. Let

$$P :\equiv \; I \xleftarrow{s} E \xrightarrow{p}_\gamma B \xrightarrow{t} J \text{ and}$$

$$Q :\equiv \; J \xleftarrow{u} F \xrightarrow{q}_\gamma C \xrightarrow{v} K$$

be polynomials in $\mathrm{Poly}\,_\gamma$. Writing $D :\equiv \; \sum_{c:C} \prod_{x:F_c} B_{u(x)}$ as in proposition 5.8.15, we have

$$\text{poly-to-span}\,(Q \circ P) = {!I} \xleftarrow{f} D \xrightarrow{g} K,$$

with

$$f(c, \alpha) :\equiv \Big(\sum_{x:F_c} E_{\alpha(x)}, (x, e) \mapsto s(e) \Big) \qquad \text{and} \qquad g(c, \alpha) :\equiv v(c).$$

On the other hand, composition in $\mathrm{Span}\,(\mathcal{U})_{!_\gamma}$ is given by the following composition of spans

Pulling back along $\mathrm{id}_{!!I}$ does not change the morphism, so that this is the same as the composition of the spans

This composition is obtained by computing the following pullback:

$$!B \times_{!J} C \equiv \sum_{X:\mathcal{V}} \sum_{\pi:X\to B} \sum_{c:C} (X, t\circ\pi) =_{!J} \left(F_c, u|_{F_c}\right)$$

$$\simeq \sum_{c:C} \sum_{X:\mathcal{V}} \sum_{k:X\simeq F_c} \sum_{\pi:X\to B} t\circ\pi = u|_{F_c}\circ k \qquad \text{(equality in } \Sigma\text{-types and univalence)}$$

$$\simeq \sum_{c:C} \sum_{\pi:F_c\to B} t\circ\pi = u|_{F_c} \qquad \text{(contracting } (X,k) \text{ onto } (F_c, \text{id}))$$

$$\simeq \sum_{c:C} \sum_{\pi:F_c\to B} \prod_{x:F_c} t(\pi(x)) = u(x) \qquad \text{(function extensionality)}$$

$$\simeq \sum_{c:C} \prod_{x:F_c} \sum_{b:B} t(b) = u(x) \qquad \text{(swapping } \Pi \text{ and } \Sigma)$$

$$\equiv \sum_{c:C} \prod_{x:F_c} B_{u(x)} \qquad \text{(def. of } B_{u(x)} \equiv \text{fib}_t\, u(x))$$

$$\equiv D$$

The inverse map $l : D \to \, !B \times_{!J} C$ maps the pair (c, α) to the tuple

$$((F_c, \pi_1 \circ \alpha), c, (\text{refl}_{F_c}, \pi_2 \circ \alpha)).$$

What remains to show is that the two following triangles commute.

Let $(c, \alpha) : D$. We have

$$\mu(!\bar{s}(\pi_1(l(\alpha, c)))) = \mu(!\bar{s}(F_c, \alpha))$$
$$= \mu(F_c, \bar{s}\circ\alpha)$$
$$= \mu\left(F_c, x \mapsto \left(E_{\alpha(x), s|_{E_{\alpha(x)}}}\right)\right)$$
$$= \left(\sum_{x:F_c} E_{\alpha(x)}, (x, e) \mapsto s(e)\right)$$
$$= f(c, \alpha)$$

and $v(\pi_2(l(c, \alpha))) = v(c) = g(c, \alpha)$, which completes the proof. $\qquad \square$

5.8.3 Polynomials vs generalized species

In section 3.11.1, we mentioned how the generalized species of [Fio+07], and more specifically their extensional counterpart, the analytic functors could be seen as a generalization of Girard's normal functors [Gir88]. Using polynomials, we can give a modern definition of these normal functors:

Definition 5.8.18. Let $P = I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$ be a diagram of sets and functions, such that p has finite fibers. Then the induced polynomial functor

$$(\!|P|\!) : \text{Set}^I \to Set^J$$

is called a *normal functor*.

In other words, the normal functors are the polynomial functors induced by finitary polynomials in sets.

Working in homotopy type theory, we can make the same definition, replacing Set with hSet, and noticing that when P is a polynomial in hSet, the induced polynomial functor

$$\langle\!\langle P\rangle\!\rangle : \mathcal{U}^I \to \mathcal{U}^J$$

restricts to

$$\langle\!\langle P\rangle\!\rangle : \text{hSet}^I \to \text{hSet}^J.$$

Example 5.8.19. The list functor $X \mapsto \sum_{n:\mathbb{N}} X^n$ is a normal functor, while the multiset functor Mul is not, since its expression

$$\text{Mul}(X) = \sum_{n \in \mathbb{N}} X^n / \mathfrak{S}_n$$

involves *quotients*.

On the other hand, analytic functors $\text{Set}^I \to \text{Set}^J$ are allowed to use quotients by actions of symmetric groups in their representation. Transposing the definition of analytic functors in homotopy type theory, we see that:

Proposition 5.8.20. *A functor $F :$ hSet $\to$ hSet is analytic if and only if it is isomorphic to a functor of the form:*

$$X \mapsto \sum_{n:\mathbb{N}} (A_n \times X^n)/\mathfrak{S}_n,$$

where for every $n : \mathbb{N}$, A_n is a set equipped with an action of $\mathfrak{S}_n$.

Remark 5.8.21. The data of the sets A_n together with their group actions is traditionally represented as a functor $A : \text{Fin} \to \text{hSet}$, this is the original notion of *combinatorial species* due to Joyal [Joy81].

It turns out that polynomials in 1-types can also give rise to this kind of expression: consider a set X equipped with an action of $\mathfrak{S}_n$. The data of such an action is equivalent to the data of a map

$$F : (\sum_{E:\mathcal{U}} \|E = \text{Fin}_n\|_{-1}) \to \mathcal{U}$$

such that $f(\text{Fin}_n, \text{refl}) = X$.

Definition 5.8.22. We write $B\mathfrak{S}_n :\equiv \sum_{E:\mathcal{U}} \|E = \text{Fin}_n\|_{-1}$. $B\mathfrak{S}_n$ is called the *classifying space* of $\mathfrak{S}_n$.

Lemma 5.8.23. *Let $F : B\mathfrak{S}_n \to \mathcal{U}$ be an action of $\mathfrak{S}_n$ on a set $X :\equiv F(\text{Fin}_n, \text{refl})$. There is a canonical equivalence of types*

$$X/\mathfrak{S}_n \simeq \| \sum_{E:B\mathfrak{S}_n} F(E)\|_0.$$

In other words, set-theoretic quotients can be recovered as set-truncations of dependent sums over 1-types.

Proof. We haven't explicitly defined $X/\mathfrak{S}_n$ in homotopy type theory, but it is a straightforward exercise to show that $\| \sum_{E:B\mathfrak{S}_n} F(E)\|_0$ satisfies the universal property of the quotient in hSet using fact 4.5.6 and the elimination principle for set truncation. $\square$

Example 5.8.24. We have $\| \text{HMul}(X)\|_0 \simeq \text{Mul}(X)$.

As a more general corollary of lemma 5.8.23, we have:

Corollary 5.8.25. *Let $A : \mathrm{Fin} \to \mathrm{hSet}$ be a combinatorial species. Writing $A_n :\equiv A(\mathrm{Fin}_n)$, we have*

$$\sum_{n:\mathbb{N}} (A_n \times X^n)/\mathfrak{S}_n \simeq \| \sum_{E:\mathrm{Fin}} A(E) \times X^E \|_0.$$

In particular, writing P for the polynomial

$$1 \leftarrow \sum_{E:\mathrm{Fin}} A(E) \times E \xrightarrow{(E,a,e) \mapsto (E,a)} \sum_{E:\mathrm{Fin}} A(E) \to 1,$$

we have

$$\sum_{n:\mathbb{N}} (A_n \times X^n)/\mathfrak{S}_n \simeq \| (\!| P |\!)(X) \|_0.$$

This result can be extended to more general I and J:

Theorem 5.8.26. *Let I and J be 1-types, $A : !_{\mathrm{Fin}}(I) \times J \to \mathrm{hSet}$ be a generalized species, and let $F : \mathrm{hSet}^I \to \mathrm{hSet}^J$ denote the induced analytic functor. Write P for the polynomial*

$$I \xleftarrow{p(e) \leftarrow(((E,p),j),a,e)} \sum_{j:J} \sum_{(E,p):!_{\mathrm{Fin}} I} A((E,p),j) \times E \xrightarrow{(E,a,e) \mapsto (E,a)} \sum_{j:J} \sum_{(E,p):!_{\mathrm{Fin}} I} A((E,p),j) \xrightarrow{\pi_1} J$$

Then for every $X : \mathrm{hSet}^I$, we have

$$F(X) =_{\mathrm{hSet}^J} \| (\!| P |\!)(X) \|_0.$$

In other words:

Every analytic functor can be recovered as the set truncation of a finitary polynomial functor in 1-types.

From this point of view, the Seely wild category of theorem 5.7.33 can be seen as a non-truncated variant of the model of linear logic of profunctors and generalized species described in [FGH24]. Moreover, we will see in section 5.9 how our model also generalizes the differential structure of the model of [FGH24] (theorem 5.9.10 and remark 5.9.15).

Theorem 5.8.26 can also give us a deeper insight into why the monad Mul on Set cannot be lifted to a comonad on $\mathrm{Span}(\mathrm{Set})$, but the HMul monad on $\mathcal{U}$ can be lifted to a comonad on $\mathrm{Span}(\mathcal{U})$. In [Fio14], Fiore shows that analytic functors

$$\mathrm{hSet}^I \to \mathrm{hSet}^J$$

with I and J groupoids (i.e. in our setting, 1-types) can be characterized as functors that preserve *filtered colimits* and *wide quasi-pullbacks*.

Definition 5.8.27. A commutative square

$$\begin{array}{ccc} u & \longrightarrow & y \\ \downarrow & & \downarrow \\ x & \longrightarrow & z \end{array}$$

in a wild category $\mathcal{C}$ with pullbacks is a *quasi-pullback* if the canonical map

$$u \to x \times_z y$$

is an epimorphism in $\mathcal{C}$.

Wide pullbacks are limits of diagrams of the form
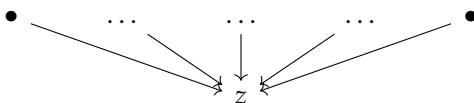
where the input arrows to z are indexed by a small set, and *wide quasi-pullbacks* are the counterpart to the previous definition for wide pullbacks.

On the other hand, polynomial functors have been characterized by Joachim Kock as functors that preserve wide pullbacks, and the finitary polynomial functors as those that additionally preserve filtered colimits [Koc09, pp. 8.6.3–4]. Additionally, it can be shown in the ∞-categorical setting that the truncation functor

$$\| - \|_n : \mathcal{U} \to \mathcal{U}$$

preserves filtered colimits.

As regard to wide pullbacks, $\| - \|_n$ does not preserve pullbacks (see remark 5.4.7), but it does turn pullbacks into quasi-pullbacks. Combining these few facts, we get a purely extensional proof that the truncation of finitary polynomial functors are analytic functors.

Remark 5.8.28. In presheaf categories hSet^I, every epimorphism is a regular epimorphism. In particular analytic functors $\mathrm{hSet}^I \to \mathrm{hSet}^J$ are relational functors. This gives an abstract interpretation of why Mul : Set $\to$ Set can be lifted to Rel $\to$ Rel : it is the set truncation of the finitary polynomial functor in 1-types HMul, and hence an analytic functor (example 5.8.24).

Going back to the problem we discussed in section 3.7 of lifting Mul to a functor Span (Set) $\to$ Span (Set), we see that Mul does not preserve pullbacks because the set truncation turned its pullback preservation into a quasi-pullback preservation. In order to recover an actual pullback preservation, we need to get rid of $\| - \|_0$, and in order to do that we have to work with HMul, leaving the realm of sets to enter the realm of 1-types, i.e. groupoids.

We close this section by noting that the theory of polynomial functors has been extended to the ∞-categorical setting by Gepner, Haugseng and Kock in [GHK22], where they also point out that the notion of analytic functors in sets is flawed due to the bad behaviour of quotients in hSet rather than in $\mathcal{U}$ [GHK22, Rmk 3.2.11], and that the right generalization of analytic functors to the ∞-categorical setting coincides with the ∞-categorical notion of finitary polynomial functors.

5.9 Spans as a model of DiLL

In section 5.8.3, we saw how finitary polynomials in homotopy type theory could be seen as a further generalization of generalized species of structure. The model of linear logic based on profunctors and generalized species described in [FGH24] is not only a model of ILL, but also of differential linear logic. In this section, we explore the differential structure our models of spans and polynomials, and give a necessary and sufficient condition on the universe of small types $\mathcal{V}$ for it to define a (wild) bicategorical model of linear logic in the sense of [FGH24]. The content of this section is mostly computational, the main result being contained in theorem 5.9.10.

### 5.9.1 A wild *bi*categorical model of linear logic

To investigate the differential structure of the wild category Span $(\mathcal{U})$, we rely on the axiomatization of [FGH24] for models of differential linear logic in bicategories. In order to do that we first need to define the bicategorical structure on Span $(\mathcal{U})$, or more generally on Span $(\mathcal{C})$. In

this section, we assume the reader has some degree of familiarity with the theory of bicategories, and we do not give an extensive survey of the theory of wild bicategories.

Definition 5.9.1. A wild (pre)bicategory is a wild category together with the usual bicategorical structures. In other words the definition of wild (pre)bicategories is the classical definition of bicategories, but with types in homotopy type theory instead of sets in set theory.

In a wild bicategory, we can in particular define adjunctions between morphisms, and if $f : x \to y$ is left adjoint to $g : y \to x$ in $\mathcal{C}$ we write $f \dashv g$.

Proposition 5.9.2. *Let $\mathcal{C}$ be a wild category with pullbacks. Then* $\mathrm{Span}\,(\mathcal{C})$ *can be endowed with the structure of a wild bicategory, where the 2-morphisms are given by commutative diagrams*
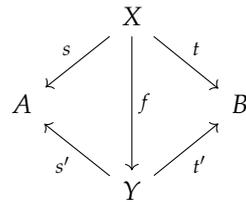
or in other words,

$$\mathrm{Span}_{\mathcal{C}}(a,b)((x,s,t),(y,s',t')) :\equiv \sum_{f:x\to y} (s' \circ f = s) \times (t' \circ f = t).$$

The functoriality of the composition of spans with respect to 2-morphisms is defined using the universal property of pullbacks, see example 5.9.4.

Alternatively, when $\mathcal{C}$ has finite products, the hom-category $\mathrm{Span}_{\mathcal{C}}(a,b)$ can just be defined to be the slice wild category $\mathcal{C}_{/a\times b}$.

Notation 5.9.3. When $f,g : X \to Y$ are functions between types, and $p : f = g$, and $x : X$, we write
$$p_x : f(x) = g(x)$$
the path defined by $p_x :\equiv \ \mathrm{ap}_{h\mapsto h(x)}(p)$.

Example 5.9.4. Let's write whiskering in detail in the wild bicategory $\mathrm{Span}\,(\mathcal{U})$. Given a diagram as follows :
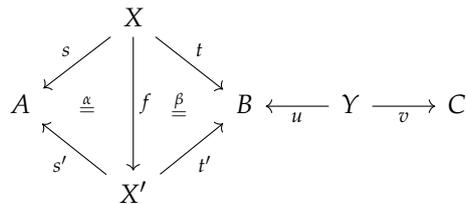
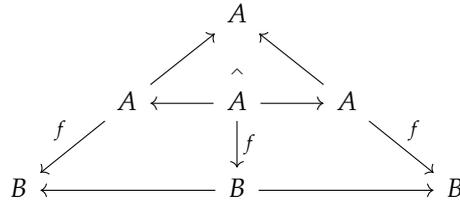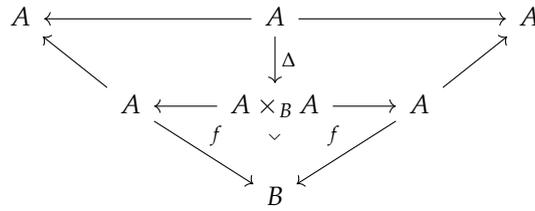the whiskering of f and (Y,u,v) is given by
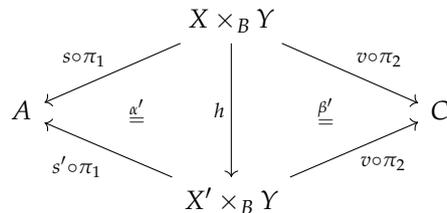
with

$$h(x, y, \gamma : t(x) = u(y)) :\equiv (f(x), y, \beta_x \cdot \gamma : t'(f(x)) = u(y))$$
$$\alpha'(x, y, \gamma) :\equiv \alpha_x$$
$$\beta'(x, y, \gamma) :\equiv \mathrm{refl}_{v(y)}$$

Proposition 5.9.5. *Let* $f : x \to y$ *be a morphism in a wild category* $\mathcal{C}$ *with pullbacks. Then* $\lambda f \dashv \rho f$ *in the wild bicategory* $\mathrm{Span}\,(\mathcal{C})$.

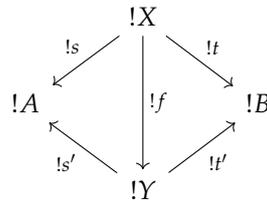*The unit* η *is given by the diagonal map* $A \to A \times_B A$:

and the counit ε *is given by the map* $f : A \to B$:

Proposition 5.9.6. *The comonad* $!_\mathcal{V}$ *on the wild category* $\mathrm{Span}\,(\mathcal{U})$ *extends to the bicategorical structure.*

Proof. Indeed, given a morphism of spans $f : (X, s, t) \to (Y, s', t')$ as a above, $!f$ is given by

which commutes by functoriality of $! : \mathcal{U} \to \mathcal{U}$. The other coherences follow from straightforward computations in spans of types. $\square$

5.9.2 The axiomatization of bicategorical DiLL

The axiomatization we follow for bicategorical models of differential linear logic is detailed in section 7 of [FGH24]. Observing that the wild bicategory $\mathrm{Span}\,(\mathcal{U})$ admits biproducts (by proposition 5.6.19 and remark 5.6.21), we want to show that our comonad $!_\mathcal{V}$ on the wild bicategory $\mathrm{Span}\,(\mathcal{U})$ admits a *codereliction*, in the sense of [FGH24, Def 7.5]. First, we show our model satisfies hypothesis 7.1 of [FGH24], which we recall word for word below:

Hypothesis 5.9.7.   1. For each $A, B \in \mathcal{K}$ the convolution monoidal structure on $\mathcal{K}[A, B]$ obtained by the biproduct structure, is cocartesian, i.e. that for $f : A \to B$ and $g : A \to B$, their convolution product $f + g : A \to B$, as defined in [...], is a coproduct of f and g in the category $K[A, B]$ and that the zero map $0_{A,B} : A \to B$, as defined in [...], is an initial object of $K[A, B]$.

2. For every $A \in \mathcal{K}$, the component of the counit $\mathsf{d}_A : {!A} \to A$ of the pseudocomonad has a left adjoint $\bar{\mathsf{d}}_A : A \to {!A}$ in $\mathcal{K}$, so that we have a unit η and counit ε, as in

$$
\begin{array}{ccc}
A \xrightarrow{\ \bar{\mathsf{d}}_A\ } {!A} & \qquad & {!A} \xrightarrow{\ \mathsf{d}_A\ } A \\
\ \ {\scriptstyle \eta_A}\nearrow \ \ \downarrow{\scriptstyle \mathsf{d}_A} & & \ \ {\scriptstyle \varepsilon_A}\swarrow \ \ \downarrow{\scriptstyle \bar{\mathsf{d}}_A} \\
{\scriptstyle 1_A}\searrow \ \ A & & {\scriptstyle 1_{!A}}\searrow \ \ {!A}
\end{array}
$$

In their notation, $\mathcal{K}$ is the linear (bi)category with biproducts, d_A is the counit of the comonad $!$, called the *dereliction* $\mathsf{d}_A : {!A} \nrightarrow A$. The multiplication of the comonad is written p_A (p for *promotion*, as it corresponds to the promotion rule of linear logic). In our setting, these maps are respectively what we called $R\eta_A$ and $R\mu_A$, where η and μ are the monad laws for $! : \mathcal{U} \to \mathcal{U}$. Since the following discussion is quite notation-heavy, we follow the notations of [FGH24] to avoid confusion, renaming our morphisms as follows:

$$
\begin{aligned}
\delta_A &: A \to {!A} & \delta_A(a) &:\equiv (1, \mathsf{cst}_A) \\
\pi_A &: {!!A} \to {!A} & \pi_A(E, F, p) &:\equiv (\Sigma EF, p) \\
\upsilon_A &: 1 \to {!A} & \upsilon_A(\star) &:\equiv (\varnothing, [\,]) \\
\gamma_A &: {!A} \times {!A} \to {!A} & \gamma_A((E, p), (F, q)) &:\equiv (E \sqcup F, [p, q])
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{d}_A &:\equiv \rho(\delta_A) : {!A} \nrightarrow A \\
\mathsf{p}_A &:\equiv \rho(\pi_A) : {!A} \nrightarrow {!!A} \\
\mathsf{w}_A &:\equiv \rho(\upsilon_A) : {!A} \nrightarrow I \\
\mathsf{c}_A &:\equiv \rho(\gamma_A) : {!A} \times {!A} \nrightarrow {!A}
\end{aligned}
$$

By proposition 5.9.5, d_A has a left adjoint $\bar{\mathsf{d}}_A :\equiv \lambda(\delta_A)$, so the item 2 of hypothesis 5.9.7 is satisfied. To prove item 1 , we first expand the definition of convolution:

Definition 5.9.8. In a wild (bi)category $\mathcal{K}$ with biproducts, given two objects A, B and two morphisms $f, g : \mathcal{K}(A, B)$, their *convolution* $f + g : \mathcal{K}(A, B)$ is defined as the composite
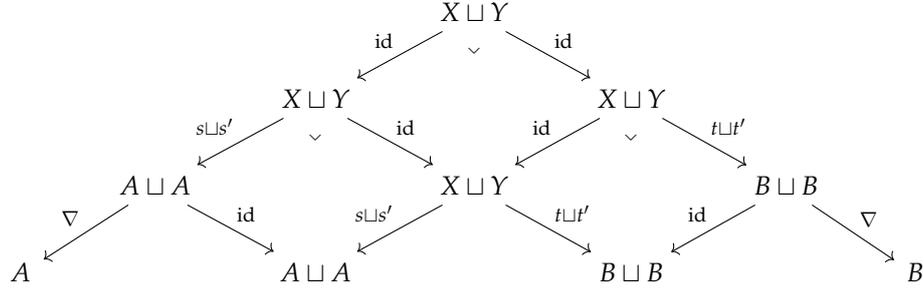
$$
A \xrightarrow{\ \Delta\ } A \oplus A \xrightarrow{\ f \oplus g\ } B \oplus B \xrightarrow{\ \nabla\ } B
$$

Where $A \oplus B$ is the biproduct of A and B, Δ is the diagonal of the product and ∇ the codiagonal of the coproduct.
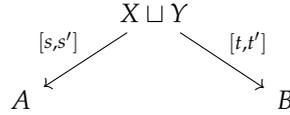
In the case of spans in $\mathcal{U}$, $A \oplus B$ is the type $A \sqcup B$, and $f \oplus g$ is the span obtained by the functoriality of $\sqcup$. In more detail, the convolution of $f :\equiv (X, s, t)$ and $g :\equiv (Y, s', t')$ is given by the composition of the following diagram of spans :

$$
\begin{array}{ccccccc}
& A \sqcup A & & X \sqcup Y & & B \sqcup B & \\
{\scriptstyle \nabla}\swarrow \quad \searrow{\scriptstyle \mathrm{id}} & & {\scriptstyle s \sqcup s'}\swarrow \quad \searrow{\scriptstyle t \sqcup t'} & & {\scriptstyle \mathrm{id}}\swarrow \quad \searrow{\scriptstyle \nabla} & \\
A & & A \sqcup A & & B \sqcup B & & B
\end{array}
$$

which we can compute in the following way :

$$
\begin{array}{c}
X \sqcup Y \\
\text{id} \swarrow \quad \vee \quad \searrow \text{id} \\
X \sqcup Y \qquad\qquad X \sqcup Y \\
s\sqcup s' \swarrow \;\; \vee \;\; \searrow \text{id} \qquad \text{id} \swarrow \;\; \vee \;\; \searrow t\sqcup t' \\
A \sqcup A \qquad\quad X \sqcup Y \qquad\quad B \sqcup B \\
\nabla \swarrow \;\text{id}\searrow \quad s\sqcup s' \swarrow \; t\sqcup t' \searrow \;\; \text{id} \swarrow \; \nabla\searrow \\
A \qquad A \sqcup A \qquad B \sqcup B \qquad B
\end{array}
$$

which reduces to

$$
\begin{array}{c}
X \sqcup Y \\
[s,s'] \swarrow \qquad \searrow [t,t'] \\
A \qquad\qquad B
\end{array}
$$

which is indeed the coproduct of f and g in the wild category $\mathrm{Span}_{\mathcal{U}}(A, B)$.

Now that we have hypothesis 5.9.7, we can show our model admits a codereliction in the sense of [FGH24, Definition 7.5] by showing three axioms are satisfied, each corresponding to the invertibility of a specific family of 2-morphisms.

5.9.3 The first comonad axiom

Firstly, the *first comonad constraint*. It is the unit for the adjunction $\bar{\mathsf{d}}_A \dashv \mathsf{d}_A$. Unfolding definitions, it's the following morphism of spans:

$$
\begin{array}{c}
A \\
\text{id}_A \swarrow \qquad \searrow \text{id}_A \\
A \quad \overset{\text{refl}}{=} \quad \eta_A \downarrow \quad \overset{\text{refl}}{=} \quad A \\
\pi_1 \searrow \qquad \swarrow \pi_2 \\
A \times_{!A} A
\end{array}
$$
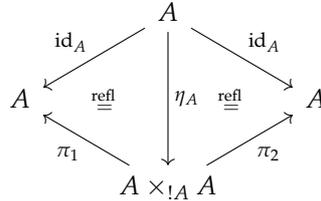
with $\eta_A(a) :\equiv (a, a, \mathrm{refl}_{(1,\mathrm{cst}_a)})$. The fact that η_A is an equivalence can be seen in multiple ways.

Let's construct an inverse explicitly. Let $(a, b, \gamma) : A \times_{!A} A$. From $\gamma : (1, \mathrm{cst}_a) = (1, \mathrm{cst}_b)$ and the fact that the type 1 has for only automorphism the identity, we deduce that $\mathrm{cst}_a = \mathrm{cst}_b$, and thus that $a = b$. Thus, the two projections $\pi_1, \pi_2 : A \times_{!A} A \to A$ are equal, both of them are inverses to η_A.

A more abstract argument would be to notice that δ_A is an embedding. Embeddings are (-1)-truncated maps of types, in the sense that their fibers are (-1)-types. One of the equivalent definitions of a morphism being n-truncated is that its diagonal is $(n-1)$-truncated, with base case $n = -2$ where (-2)-truncated means being an equivalence (by fact 4.9.5) Since the map η_A is by definition the diagonal of the map δ_A, δ_A being an embedding directly shows that η_A is an equivalence.

5.9.4 The second comonad axiom

Secondly, the *second comonad constraint*

$$
\begin{array}{ccccc}
A \times I & \xrightarrow{L\pi_1} & A & \xrightarrow{\bar{\mathsf{d}}_A} & !A \\
{\scriptstyle \bar{\mathsf{d}}_A \times \bar{\mathsf{w}}_A} \downarrow & & {\scriptstyle \mu_A} \Downarrow & & \downarrow {\scriptstyle \mathsf{p}_A} \\
!A \times !A & \xrightarrow[\bar{\mathsf{d}}_{!A} \times \mathsf{p}_A]{} & !!A \times !!A & \xrightarrow[\bar{\mathsf{c}}_{!A}]{} & !!A
\end{array}
$$

is defined to be the composite of the bicategorical diagram of 2-morphisms shown in fig. 5.1.
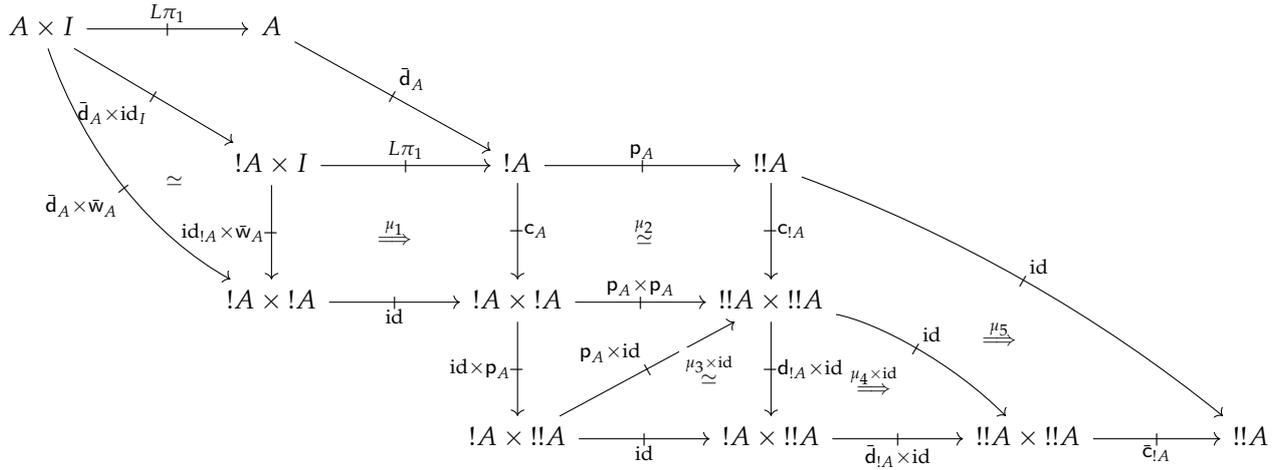
Figure 5.1: The second comonad constraint.

Unfolding all spans, naming the unnamed squares, turning the μ_1 square into a triangle to simplify things somewhat, and flipping the whole thing diagonally so it fits within a single page, we get the diagram of spans and span morphisms shown in fig. 5.2, where each span's input leg has been colored blue and their output leg red for legibility.

We write $(f_i, \alpha_i, \beta_i) :\equiv \mu_i$. We implicitly use the equality $\lambda f \circ \lambda g = \lambda(f \circ g)$ to simplify pullback computations as much as possible (and similarly for $\rho f \circ \rho g = \rho(g \circ f)$).

The 2-morphism μ_1 in the following diagram
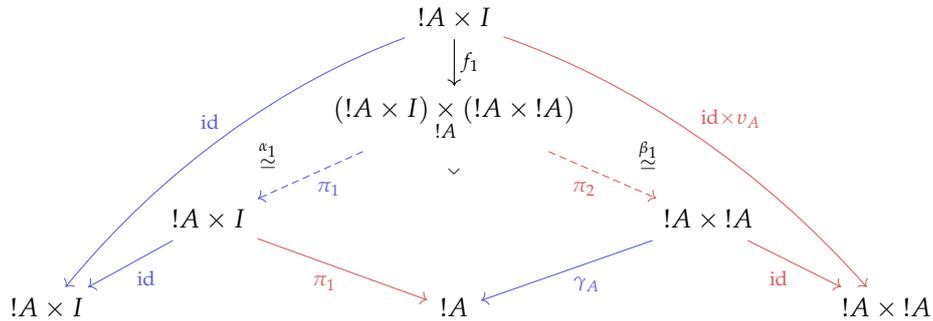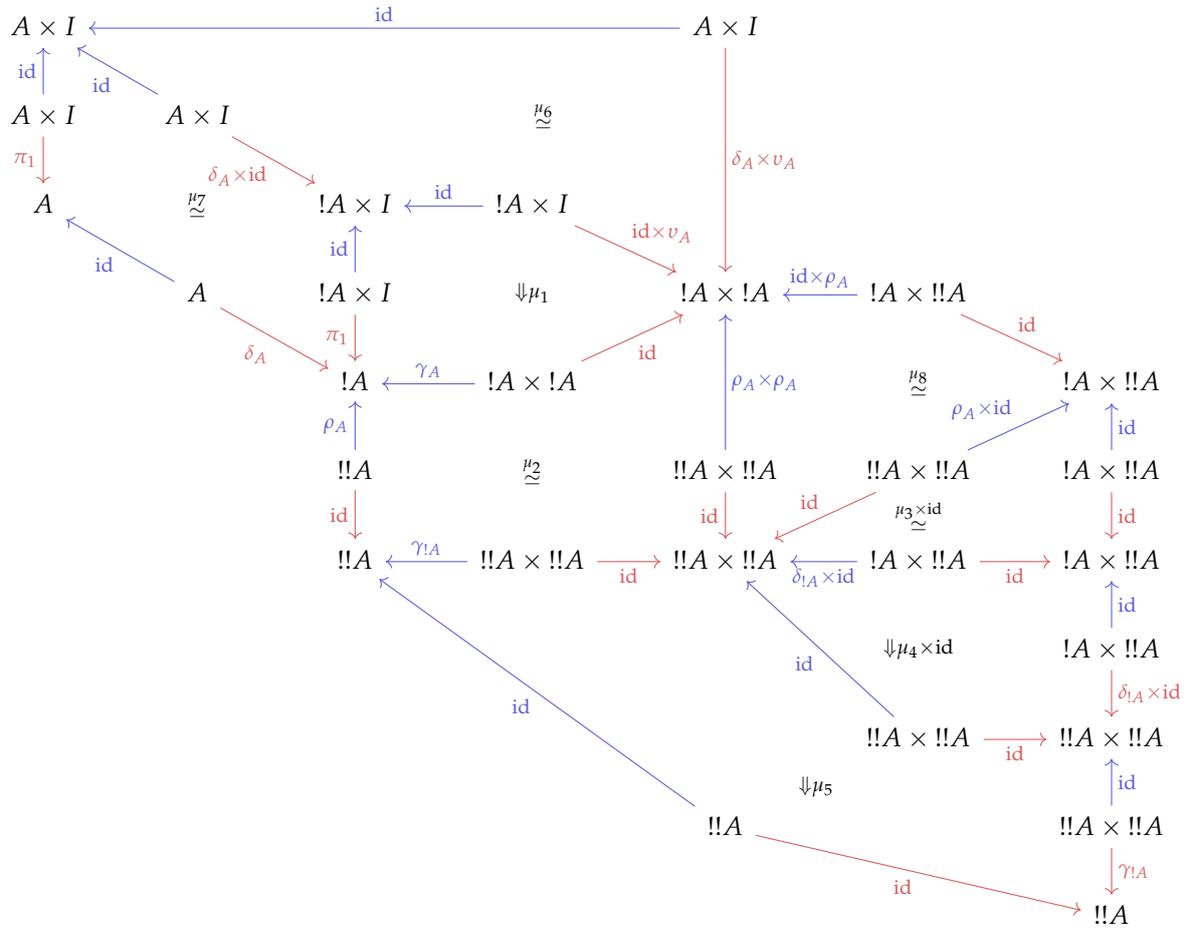
Figure 5.2: The second comonad constraint for spans. Input legs are shown in blue and output legs in red.
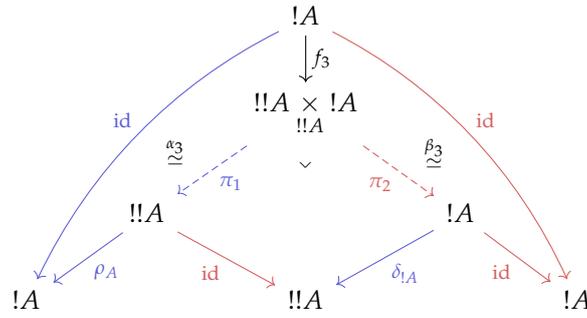
is given by

$$
\begin{aligned}
f_1((E, p), \star) &:\equiv ((E, p), (\varnothing, [])) \\
\alpha_1 &:\equiv \operatorname{refl}_{\pi_1} \\
\beta_1 &:\equiv \operatorname{refl}_{v_A \circ \pi_2} \qquad\qquad (\text{since } v_A(\star) :\equiv (\varnothing, []))
\end{aligned}
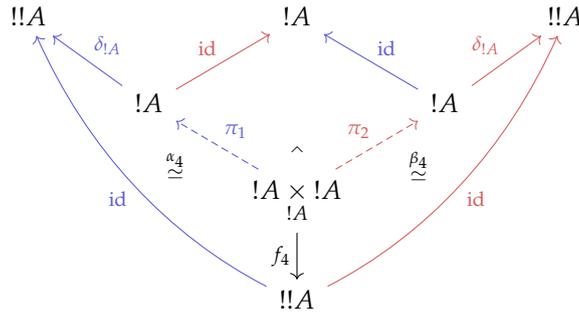$$

The 2-morphism μ_2 is given by the following diagram

$$
\begin{array}{c}
!A \xleftarrow{\quad \gamma_A \quad} !A \times !A \xrightarrow{\quad \text{id} \quad} !A \times !A \\[4pt]
\rho_A \uparrow \qquad \overset{\alpha_2}{\simeq} \qquad \pi_1 \uparrow \qquad \qquad \uparrow \rho_A \times \rho_A \\[4pt]
\qquad (!A \times !A) \underset{!A \times !A}{\times} (!!A \times !!A) \xdashrightarrow{\pi_2} !!A \times !!A \\[4pt]
\xrightarrow{f_2} \\[4pt]
!!A \xdashleftarrow{\pi_1} !!A \underset{!!A}{\times} (!!A \times !!A) \qquad \overset{\beta_2}{\simeq} \qquad \text{id} \\[4pt]
\text{id} \downarrow \qquad\qquad \downarrow \pi_2 \\[4pt]
!!A \xleftarrow{\gamma_{!A}} !!A \times !!A \xrightarrow{\qquad \text{id} \qquad} !!A \times !!A
\end{array}
$$
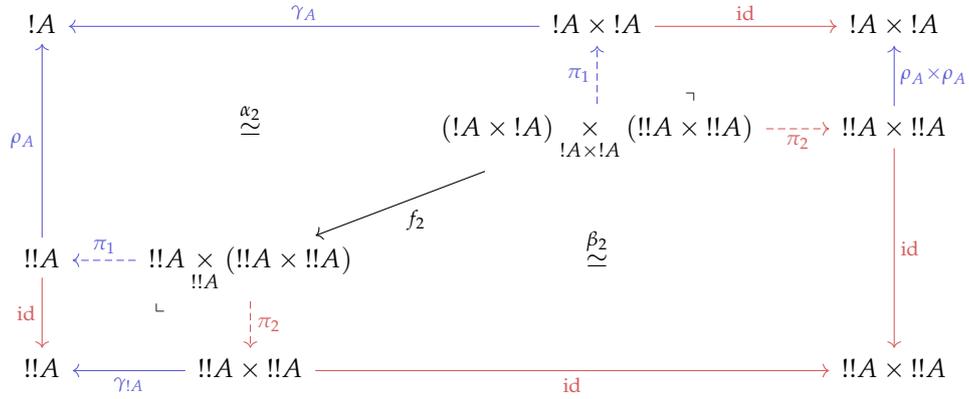
with $f_2(l, r, p) :\equiv (\gamma_{!A}(r), r, \text{refl})$, $\beta_2 :\equiv \text{refl}$ and $\alpha_{2,(l,r,p)}$ is a combination of $p : (\pi_A \times \pi_A)(r) = l$ and of the naturality square for γ for the morphism π_A.

The 2-morphism μ_3 is given by

$$
\begin{array}{c}
!A \\
\downarrow f_3 \\
!!A \underset{!!A}{\times} !A \\
\text{id} \quad \overset{\alpha_3}{\simeq} \quad \pi_1 \qquad \pi_2 \quad \overset{\beta_3}{\simeq} \quad \text{id} \\
!!A \qquad\qquad\qquad !A \\
\rho_A \qquad \text{id} \qquad \delta_{!A} \qquad \text{id} \\
!A \qquad\qquad !!A \qquad\qquad !A
\end{array}
$$

with $f_3(e) :\equiv (\delta_{!A}(e), e, \text{refl})$, $\beta_3 :\equiv \text{refl}_{\text{id}}$ and α_3 is the left unit law for the monad $(!, \rho, \delta)$.

The 2-morphism μ_4 is the counit of the adjunction $\bar{\mathsf{d}}_A \vdash \mathsf{d}_A$, or more explicitly

$$
\begin{array}{c}
!!A \qquad\qquad !A \qquad\qquad !!A \\
\delta_{!A} \quad \text{id} \qquad \text{id} \quad \delta_{!A} \\
!A \qquad\qquad\qquad !A \\
\overset{\alpha_4}{\simeq} \quad \pi_1 \qquad \pi_2 \quad \overset{\beta_4}{\simeq} \\
\text{id} \qquad !A \underset{!A}{\times} !A \qquad \text{id} \\
\downarrow f_4 \\
!!A
\end{array}
$$

with $f_4(l, r, p) :\equiv \delta_{!A}(l)$, and α_4 is reflexivity and $\beta_{4,(l,r,p)}$ is given by the application of $\delta_{!A}$ to the equality p.
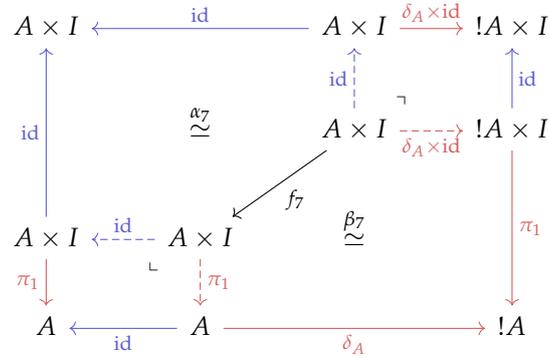
The 2-morphism μ_5 is given by
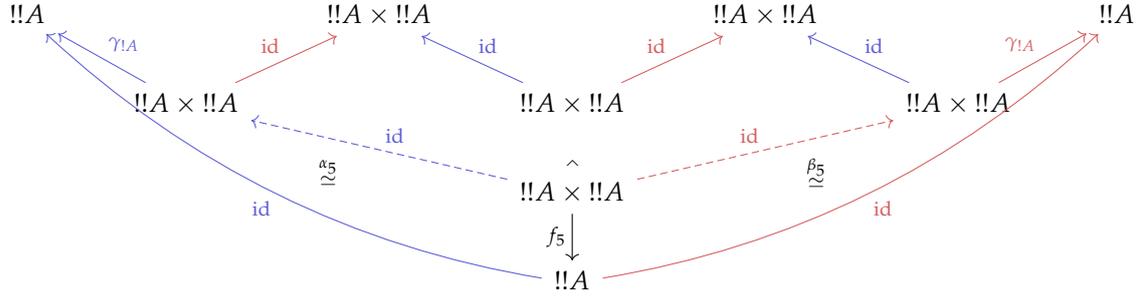
with $f_5 :\equiv \gamma_{!A}$ and α_5 and β_5 are once again reflexivity.

The equivalences μ_6 and μ_8 stem from the functoriality of the monoidal product $- \times -$ on Span $(\mathcal{U})$. For instance, μ_6 is given by
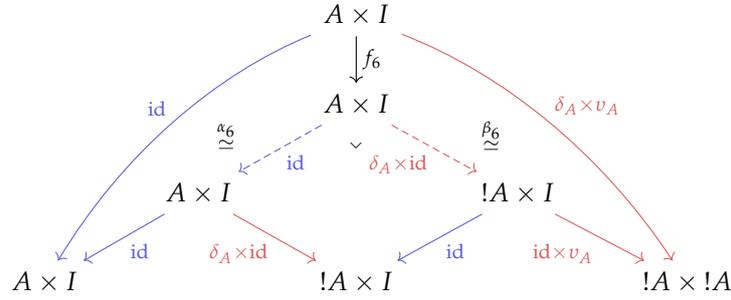
with $f_6 :\equiv \mathrm{id}$, and α and β are reflexivity.

Finally, μ_7 is straightforwardly given by

with $f_7 :\equiv \mathrm{id}$ and α and β being reflexivity once again.

With all these definitions done, the total composition of fig. 5.2 is detailed in fig. 5.3.

After simplifying everything, we find that μ is equivalent to the following map of types (forgetting the equality triangles in the morphism of spans) :

$$\sum_{a:A} \sum_{(E,F,p):!!A} (\pi_A(E,F,p) = v_A(\star)) \to \sum_{a:A} \sum_{(E',F',q):!!A} (\pi_A(E',F',q) = \delta_A(a))$$

$$(a,(E,F,p),j) \mapsto (a,(1 \sqcup E, 1 \sqcup F, [a,p]),j')$$

where in more detail, j proves that $\Sigma EF = \varnothing$ and $p = []$, so that together with $1 \sqcup \varnothing = 1$ we get j'.

Figure 5.3: The composition of the second comonad constraint.

Proposition 5.9.9. *The above map is an equivalence if and only if, the map*

$$\mu' : \sum_{E:\mathcal{V}} \sum_{F:E\to\mathcal{V}} (\Sigma EF = \varnothing) \to \sum_{E':\mathcal{V}} \sum_{F':E'\to\mathcal{V}} (\Sigma E'F' = 1)$$
$$(E, F, p) \mapsto (E \sqcup 1, F \sqcup \mathrm{cst}_1, p')$$

is an equivalence.

Informally, this means in particular that for every type E' and type family $F' : E' \to \mathcal{V}$ such that $\Sigma E'F'$ is contractible, E' must be of the form $E \sqcup 1$ and F' of the form $F \sqcup \mathrm{cst}_1$ for some type E and some type family $F : E \to \mathcal{V}$. This would be reasonable enough to expect is set theory, but is actually completely false in general in homotopy type theory. For instance, let E' be the circle $\mathbb{S}^1$ and F' the helix H defined in section 4.10. Then by facts 4.7.3 and 4.10.2 we have $\Sigma E'F' = 1$ so that it lies in the image of μ', but $\mathbb{S}^1$ is not of the form $E \sqcup 1$ for any E, as it is connected (example 4.11.6).

On the other hand, in the case $\mathcal{V} :\equiv \mathrm{Fin}$, the map above is indeed an equivalence. This can be shown using the principle of finite choice and decidable equality of finite sets, two principles that can be shown in homotopy type theory without further axioms. If one were to add the axioms of choice and of decidable equality for sets of a given universe $\mathcal{U}$, then the type $\mathcal{V} :\equiv \mathrm{hSet}_{\mathcal{U}}$ of $\mathcal{U}$-small sets would also satisfy this property. But for more general types, even considering 1-types in $\mathcal{V}$ completely breaks the second comonad axiom of [FGH24].

5.9.5 The strength constraint

Finally, the *strength constraint* is a 2-morphism $\sigma_{A,B}$

$$
\begin{array}{ccc}
A \times !B & \xrightarrow{\bar{\mathsf{d}}_A \times \mathsf{id}} & !A \times !B \\
\mathsf{id} \times \mathsf{d}_B \downarrow & \xRightarrow{\sigma_{A,B}} & \downarrow m^2_{A,B} \\
A \times B & \xrightarrow[\bar{\mathsf{d}}_{A \times B}]{} & !(A \times B)
\end{array}
$$

obtained as the following composite

where $m^2_{A,B}$ and $\nu_{A,B}$ are given by the monoidality of the comonad ! on $\mathrm{Span}_{\mathcal{U}}$. More explicitly, $m^2_{A,B}$ is given by the span

$$
!A \times !B \xleftarrow{\quad m_{A,B} \quad} !(A \times B) \xrightarrow{\quad \mathsf{id} \quad} !(A \times B)
$$

$$
(E, \pi_1 \circ p), (E, \pi_2 \circ p) \longleftarrow\!\!\!\mid (E, p) \longmapsto (E, p)
$$

and ν_A, B is given by the following map of spans:

Doing all the calculations, one finds that σ is equivalent to the following map of spans σ':

with $\sigma'(a, b) :\equiv ((a, \delta_A(b)), (\delta_{A \times B}(a, b)), \text{refl})$. It turns out we can construct an explicit inverse to σ'. Let

$$((a, (E, p)), (F, q), j) : (A \times {!}B) \underset{{!}A \times {!}B}{\times} {!}(A \times B)$$

In other words, $a : A, E, F : \mathcal{V}, p : E \to B, q : F \to A \times B$, and crucially,

$$j : ((1, \text{cst}_a), (E, p)) = ((F, \pi_1 \circ q), (F, \pi_2 \circ q))$$

Using j, we immediately get $F = 1$ and $E = F$, so that $E = 1$, and $\pi_1 \circ q = \text{cst}_a$. Now knowing $E = F = 1$, we also get that $(F, q) = \delta_{A \times B}(a, b)$ where b is the image by $\pi_2 \circ q$ of the only element in F. All in all, this means that $((a, (E, p)), (F, q), j) = \sigma'(a, b)$, so that we can define $\sigma'^{-1}(\dots) :\equiv (a, b)$. Some care is needed to make this argument more formal (for instance for the fact that we get the right j this way), a direct approach would be to construct an equivalence directly using elementary arguments on Σ-types as we have done repeatedly throughout this chapter, and then showing that its underlying function is equal to σ'.

5.9.6 Conclusion

Summing up the results of the computations of the previous sections, by proposition 5.9.9 we have the following:

Theorem 5.9.10. *The wild bicategory* $\text{Span}\,(\mathcal{U})$ *with the comonad* ${!}_\mathcal{V}$ *defines a model of differential linear logic in the sense of* [FGH24] *if and only if the following map:*

$$\sum_{E:\mathcal{V}} \sum_{F:E \to \mathcal{V}} (\Sigma EF = \varnothing) \to \sum_{E':\mathcal{V}} \sum_{F':E' \to \mathcal{V}} (\Sigma E'F' = 1)$$
$$(E, F, p) \mapsto (E \sqcup 1, F \sqcup \text{cst}_1, p')$$

is an equivalence.

In particular, one cannot allow $\mathcal{V}$ to contain types that are not sets (see the example of the circle following proposition 5.9.9). From the point of view of differentiation, it is indeed unclear what it would mean to differentiate the function

$$X \mapsto X^{\mathbb{S}^1},$$

as differentiation in polynomials is typically understood through the Leibniz rule

$$\partial(X^{A \sqcup B}) = \partial(X^A) \times X^b \sqcup X^A \times \partial(X^B).$$

From it one can easily recover for instance

$$\partial(X^{1 \sqcup 1}) = X \sqcup X,$$

but the formula does not make sense when the exponent cannot be expressed as a coproduct, as is the case of the circle $\mathbb{S}^1$.

Corollary 5.9.11. *The comonad* ${!}_{\text{Fin}}$ *makes* $\text{Span}\,(\mathcal{U})$ *into a wild bicategorical model of differential linear logic.*

Corollary 5.9.12. *Assuming the axiom of choice and the axiom decidable equality for sets* ([Uni13]), *the universe of sets* hSet *yields an exponential comonad* ${!}_{\text{hSet}}$ *that defines a wild bicategorical model of differential linear logic.*

Remark 5.9.13. As noted in remark 5.7.23, the comonad $!_{\mathcal{V}}$ should be a Lafont exponential on the ∞-category of spans of types (even though this result cannot be stated in homotopy type theory). As such, the fact that $!_{\mathrm{Fin}}$ defines a model of differential linear logic could be seen as an instance of an ∞-categorical generalization of fact 3.12.3 ([Lem21]).

Remark 5.9.14. In his article [Mel19], Melliès shows that his (homotopy) bicategory of spans of groupoids (and more generally template games) defines a model of differential linear logic, with the exponential comonad given by the *free symmetric monoidal groupoid* construction. Unfolding definitions, and representing groupoids as 1-types in homotopy type theory, this coincides with our comonad $\mathrm{HMul} \equiv !_{\mathrm{Fin}}$. In that sense, corollary 5.9.11 generalizes this instance of Melliès's model, including its differential structure, to higher homotopy types. Moreover, corollary 5.9.12 shows that we can consider more extreme ways of duplicating resources while keeping a differential structure.

Remark 5.9.15. Through the comparison of section 5.8.3, corollary 5.9.11 can also be thought of as a non-truncated variant of the model of differential linear logic given by profunctors and generalized species described in [FGH24].

Chapter 6

∞-category theory

As we explained in the introduction (section 1.5), the rest of this thesis is dedicated to describing models of linear logic in the setting of ∞-category theory, in order to avoid the limitations of homotopy type theory when it comes to homotopy-coherent structures. In this chapter we review the foundations of this theory, in the formalism due to Joyal and Lurie.

We will make extensive use of references to Lurie's books [Lur09; Lur17], as well as his *Kerodon* project [Lur18]. As a brief survey of these references, *Higher topos theory* lays out the foundations of the theory: ∞-categories, functors, natural transformations, limits and colimits, adjunctions, presentable ∞-categories, as well as the very important indexed/fibred correspondence, which in this context is more often called called the straightening/unstraightening equivalence. The rest of the book is dedicated to laying out the theory of ∞-toposes, which we will not need in this thesis. The book *Higher Algebra* sets up the theory of (symmetric) monoidal structures in the ∞-categorical context, as an instance of the more general theory of ∞-operads. We will heavily rely on this theory to be able to define linear/non-linear adjunctions between ∞-categories and Lafont ∞-categories, as well as construct actual examples of such higher models of linear logic. Finally, the *Kerodon* is an evolving online book project modeled on the *Stacks project*, and it gives a somewhat different approach to the theory, relying less on the theory of model categories and working more directly in the ∞-categorical setting. Since it is constantly evolving, the *Kerodon* uses a *tag system* to have permanent links to definitions and theorems: for instance when we write [Lur18, Tag 04BH] (clickable in the pdf version of this thesis), we refer to the webpage accessed at the url `https://kerodon.net/tag/04BH`.

The theory of ∞-categories can be very technical at times, so for the most part we will not write proofs of the results we cite, relying instead on 1-categorical intuitions to not lose the reader: indeed while the formalism may be very different between 1-categories and ∞-categories, most of the ideas and results actually hold in the same way in both settings, appropriately replacing the word "set" by "∞-groupoid" where appropriate.

This chapter can be roughly divided in two parts: first the fundamental concepts of the theory, and then the theory of higher algebra. In the fundamentals, we begin in section 6.1 with the theory of simplicial sets: the fundamental objects underlying ∞-categories and ∞-groupoids in the formalism of Joyal-Lurie. Having defined simplicial sets, we go on to define ∞-categories, ∞-groupoids, functors and natural transformations in section 6.2, and study equivalences between ∞-categories in section 6.3. In section 6.4, we review the indexed/fibred correspondence, or rather correspondence**s** in the setting of ∞-categories, better known in that context as the *straightening/unstraightening equivalence*, for which we define the ∞-category ∞ Cat of ∞-categories and the ∞-category $\mathcal{S}$ of ∞-groupoids. While this correspondence is considered a pretty advanced topic

in ordinary category theory, it is absolutely fundamental in the ∞-categorical setting: for instance it is paramount to the definition of the hom functor, and hence of the Yoneda lemma, which we review in section 6.5. We close this first part with the theory of adjunctions in section 6.7 and of presentable ∞-categories in section 6.8, including the very important adjoint functor theorem (fact 6.8.15).

The second part is dedicated to *higher algebra*: everything involving homotopy-coherent structures on ∞-categories and ∞-groupoids ((symmetric) monoidal structures, monads, algebras over monads...). We introduce the general ideas in section 6.9, motivate and review Lurie's definition and theory of symmetric monoidal ∞-categories in section 6.10, to then study commutative monoids and free commutative monoids in symmetric monoidal ∞-categories in section 6.11. We then take a detour to study internal homs and closed symmetric monoidal ∞-categories in section 6.12, providing proofs for results that seem to be folklore but for which we could not find any reference in the literature. Finally, in section 6.13 we discuss Lurie's more general theory of ∞-operads to define non-symmetric monoidal categories, monads and their algebras, and we review the links between monads and adjunctions.

6.1 The formalism of simplicial sets

In homotopy type theory, ∞-groupoids were primitive objects of the formalism — every type was thought of as an ∞-groupoid, with the structure of isomorphisms and higher isomorphisms given by the iterated equality types $x = y$ — and higher categories (in the guise of wild categories) were defined internally to this homotopical setting. In set theory, the notion of ∞-groupoid is clearly not a primitive notion, and there are many possible choices of *models* for ∞-groupoids, and similarly for ∞-categories.

Remark 6.1.1. As a reminder, by ∞-category we mean $(\infty, 1)$-category, see definition 2.3.15 and example 2.3.16.

Most models rely on the idea of using *shapes* to present the higher morphisms (or higher *cells*) in a higher groupoid/category, and in our case the shapes in question are *simplices*, which are traditionally defined geometrically as follows:

Definition 6.1.2. An *n*-dimensional *geometric simplex* (or *n*-simplex) is the convex hull of $n + 1$ affinely independent points in $\mathbb{R}^k$.

For instance, a 0-simplex is a point, a 1-simplex is an interval, a 2-simplex is a triangle, a 3-simplex is a tetrahedron.

The idea is that every ∞-category $\mathcal{C}$ should have for every $n \in \mathbb{N}$ a set $\mathcal{C}_n$ of *n-dimensional cells with the shape of an n-simplex*, every *n*-cell $\sigma \in \mathcal{C}_n$ should yield $(n - 1)$-cells given by its faces, and the operation of taking faces should be coherent with the geometric structure of simplices. The data of all the $(\mathcal{C}_n)_{n \in \mathbb{N}}$ is called a *simplicial set*, and it is encoded as a functor $\Delta \to \mathrm{Set}$, where Δ is a category which represent all the combinatorial structure of simplices.

In this section we survey the foundations of the theory of simplicial sets, and what conditions a simplicial set must satisfy to be considered an ∞-category or an ∞-groupoid.

Definition 6.1.3. Given $n \in \mathbb{N}$, write $[n]$ for the linearly ordered set $\{0 < \cdots < n\}$. The *simplex category* Δ is the category whose objects are the $[n]$ for $n \in \mathbb{N}$ and whose morphisms are the (non-strictly) order-preserving maps.

For instance, the two maps $[0] \to [1]$ correspond to the two inclusions of an end-point into

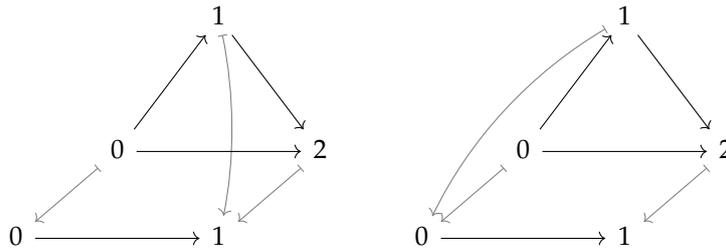the interval

$$
\begin{array}{ccccc}
[0] & 0 & & 0 & [0] \\
& \downarrow & & \downarrow & \\
[1] & 0 \longrightarrow 1 & & 0 \longrightarrow 1 & [1]
\end{array}
$$

while the three injective maps $[1] \to [2]$ correspond to the three inclusions of the interval into the triangle
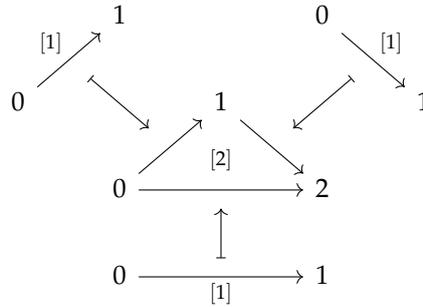
and the two surjective maps $[2] \to [1]$ correspond to the two ways of collapsing the triangle onto the interval:

More generally, the injective morphisms $[m] \to [n]$ correspond to the different ways of *affinely embedding* a geometric m-simplex as a subsimplex of a geometric n-simplex, while the surjective morphisms $[m] \to [n]$ correspond to ways in which one can *affinely collapse* a geometric m-simplex onto a geometric n-simplex. Since every morphism $f : [m] \to [n]$ factors uniquely as a surjection followed by an injection

$$
[m] \xrightarrow{\;f_{\mathrm{surj}}\;} [k] \xrightarrow{\;f_{\mathrm{inj}}\;} [n],
$$
$$
\underbrace{\phantom{[m] \xrightarrow{\;f_{\mathrm{surj}}\;} [k] \xrightarrow{\;f_{\mathrm{inj}}\;} [n]}}_{f}
$$

we can think of every morphism morphism $f : [m] \to [n]$ as collapsing a geometric m-simplex onto a subsimplex of a geometric n-simplex.

Remark 6.1.4. Any non-empty linearly ordered finite set is isomorphic as a partial order to $[n]$ for some $n \in \mathbb{N}$, because of this the category Δ is equivalent to the category Lin of non-empty linearly ordered finite sets and (non-strictly) order-preserving functions.

Definition 6.1.5. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. The *morphisms of simplicial sets* are given by natural transformations, and we write sSet for the category of simplicial sets.

More generally, when $\mathcal{C}$ is a category, the functors

$$
X : \Delta^{\mathrm{op}} \to \mathcal{C}
$$

are called *simplicial objects* in $\mathcal{C}$, and the functors

$$X : \Delta \to \mathcal{C}$$

are called *cosimplicial object* is $\mathcal{C}$.

Notation 6.1.6. Given a simplicial set X, we write $X_n := X([n])$.

As our first example of simplicial set, we have:

Definition 6.1.7. The *standard n-simplex* Δ^n is the simplicial set

$$よ([n]) := \big([k] \mapsto \mathrm{Hom}_\Delta([k], [n])\big).$$

Fact 6.1.8. *For every simplicial set X, there is a natural bijection*

$$X_n \simeq \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X).$$

whose inverse map is given by

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \tag{6.1}$$

Proof. This is the Yoneda lemma for the category Δ. $\qquad\square$

This motivates the following terminology:

Definition 6.1.9. Let $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The elements of X_0 are called the *vertices* of X, the elements of X_1 are called the *edges* of X, the elements of X_2 are called the *triangles* in X, and for general $n \in \mathbb{N}$ the elements of X_n are called *n-simplices* in X.

To accent the intuition that simplicial sets can be used to represent (higher) categorical structures, we will also call vertices "*objects*", edges "*morphisms*", and triangles "*commutative triangles*".

Definition 6.1.10 ([Lur18, Tag 04ZB]). Let X be a simplicial set, and $0 \le i \le n$, with $n \ge 1$. The action of X on the morphism $\delta_i^n : [n-1] \to [n]$ given by

$$\delta_i^n(j) := \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \ge i \end{cases}$$

determines a map $d_i^n : X_n \to X_{n-1}$, or d_i for short, called the *i-th face map*.

Since the map δ_i^n is also the unique injective morphism $[n-1] \to [n]$ whose image does not contain i, the map $d_i : X_n \to X_{n-1}$ should be thought of as mapping an n-simplex in X to its $(n-1)$-dimensional subsimplex that does not contain the vertex i.

Example 6.1.11. Let $f \in X_1$ be an edge in X. The vertex $d_1(f)$ is the *source vertex* of x, while $d_0(f)$ is its *target vertex*, so that we may write:

$$d_1(f) \xrightarrow{\ f\ } d_0(f)$$

Example 6.1.12. Let $\sigma \in X_2$ be a triangle in X, it may be drawn as:

$$x = d_1(d_2(\sigma)) = d_1(d_1(\sigma))$$
$$y = d_0(d_2(\sigma)) = d_1(d_0(\sigma))$$
$$z = d_0(d_0(\sigma)) = d_0(d_1(\sigma))$$

Definition 6.1.13 ([Lur18, Tag 04ZJ])**.** Let X be a simplicial set, and $0 \le i \le n$, with $n \ge 1$. The action of X on the morphism $\sigma_i^n : [n+1] \to [n]$ given by

$$\sigma_i^n(j) := \begin{cases} j & \text{if } j \le i \\ j-1 & \text{if } j \ge i+1 \end{cases}$$

determines a map $s_i^n : X_n \to X_{n+1}$, or s_i for short, called the *i-th degeneracy map*.

Example 6.1.14. Let $x \in X_0$ be a vertex in a simplicial set X. The edge $s_0(x) \in X$ has for source and target vertex x: it is called the *identity* of x. We write $\mathrm{id}_x := s_0(x)$:

$$x \xrightarrow{\ \mathrm{id}_x\ } x$$

Example 6.1.15. Let $f : x \to y$ be an edge in a simplicial set X. The triangles $s_0(f)$ and $s_1(f)$ may be written as

Definition 6.1.16. A simplex $\tau \in X_{n+1}$ is said to be *degenerate* if it is in the image of a degeneracy map $s_i : X_n \to X_{n+1}$ for some $0 \le i \le n$. It is said to be *non-degenerate* if it is not degenerate.

The degenerate simplices are very useful when considering morphisms of simplicial sets: they allow one to represent maps that *collapse* high-dimensional simplices to lower-dimensional ones. For instance, let X and Y be simplicial sets, and $x \in X_m$ and $y \in Y_n$, with $m > n$. A morphism of simplicial sets $f : X \to Y$ consists by definition of maps $f_k : X_k \to Y_k$, so to represent a map $f : X \to Y$ that "maps x to y", we can define f such that the map $f_m : X_m \to Y_m$ maps x to a degenerate version of y, i.e. the image of y in Y_m by the action of Y on a surjective map $[m] \to [n]$.

In terms of geometric intuition, it is better to think only of the non-degenerate simplices as having a geometric meaning, and think of the degenerate simplices as a technical convenience. For instance, when representing a simplicial set pictorially, only its non-degenerate simplices are drawn: see for instance fig. 6.1.

Just like for graphs and categories, one can reverse all the edges in a simplicial set and get another simplicial set:

Notation 6.1.17. Let $\mathrm{Op} : \Delta \to \Delta$ denote the involutive functor

$$\Delta \to \Delta$$
$$[m] \mapsto [m]$$
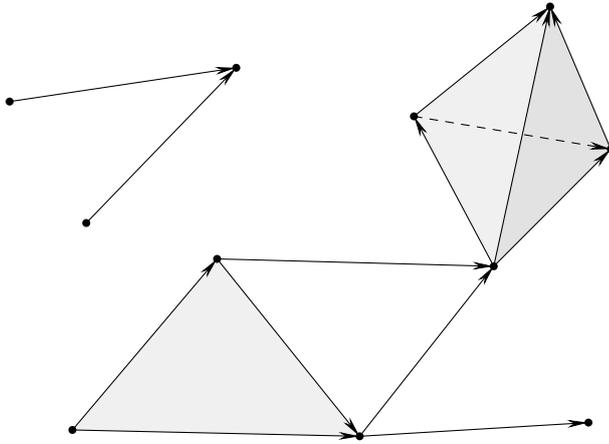$$(\alpha : [m] \to [n]) \mapsto (i \mapsto n - \alpha(m-i))$$

Figure 6.1: Drawing of a simplicial set, representing only non-degenerate simplices.

Note that under the identification of Δ with the category of non-empty finite linearly ordered sets (remark 6.1.4) Lin, Op corresponds to the involution $\mathrm{Lin} \to \mathrm{Lin}$ that maps $(E, \leq)$ to $(E, \geq)$.

Definition 6.1.18. Let $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Its *opposite simplicial set* X^{op} is defined to be the composition $X \circ \mathrm{Op}$:

$$\Delta^{\mathrm{op}} \xrightarrow{\mathrm{Op}^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathrm{Set}$$

Since the composition of functors is functorial with respect to natural transformations, for every morphism of simplicial sets $f : X \to Y$ we also get an *opposite morphism* $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$.

In section 2.2, we explained that homotopy types should correspond to ∞-groupoids (the so-called homotopy hypothesis slogan 2.2.19). In the setting of simplicial sets, this hypothesis is incarnated by a functor that associates to every topological space a simplicial set, as follows:

Notation 6.1.19. For every $n \in \mathbb{N}$, let $|\Delta^n|$ the underlying topological space of a geometric simplex (for instance the convex hull of the standard euclidean basis in $\mathbb{R}^{n+1}$).

Example 6.1.20. Let X be a topological space. The assignment

$$[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}\left(|\Delta^n|, X\right)$$

defines a simplicial set $S(X)$ called the *singular complex* associated to X, and the operation

$$X \mapsto S(X)$$

defines a functor $\mathrm{Top} \to \mathrm{sSet}$.

Similarly, we can also associate to every category a simplicial set:

Example 6.1.21. Every partial order can be regarded as a category, so the category Δ can be seen as a full subcategory of Cat. In particular, for every category $\mathcal{C}$ one can consider the simplicial set

$$\mathrm{N}\,(C) : [n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}\,([n], C),$$

called the *nerve* of C. The operation $\mathcal{C} \mapsto \mathrm{N}\,(\mathcal{C})$ defines a functor

$$\mathrm{N} : \mathrm{Cat} \to \mathrm{sSet}.$$

Fact 6.1.22. *The nerve functor* $\mathrm{N} : \mathrm{Cat} \to \mathrm{sSet}$ *is fully faithful.*

Fact 6.1.23. *There is a canonical isomorphism* $\mathrm{N}\,(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{N}\,(\mathcal{C})^{\mathrm{op}}$.

Remark 6.1.24. Examples 6.1.20 and 6.1.21 are two instances of a more general *nerve construction*: the inclusion $\Delta \hookrightarrow \mathrm{Cat}$ and the functor $[n] \mapsto |\Delta^n|$ defines cosimplicial objects in Cat and Top respectively. The data of a cosimplicial object $F : \Delta \to \mathcal{C}$ always defines a functor $\mathcal{C} \to \mathrm{sSet}$ via the formula

$$c \mapsto ([n] \mapsto \mathrm{Hom}_{\mathcal{C}}\,(F([n]), c)),$$

and this functor is usually called the *nerve functor* relative to F (also note that this is purely formal: the category Δ may be replaced with any small category and this would still work).

Of course, not every simplicial set is of the form $\mathrm{N}\,(\mathcal{C})$ or $S(X)$, as the operations of composition of morphisms in a category and composition of paths in a topological space endow $\mathrm{N}\,\mathcal{C}$ and SX with additional properties, formulated in terms of *lifting condition*.

Definition 6.1.25 ([Lur18, Tag 000K]). Let $0 \leq k \leq n$ be natural numbers, with $n > 0$. The *horn* Λ_k^n is a simplicial set $\Lambda_k^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$ defined by:

$$\Lambda_k^n([m]) := \{\alpha \in \mathrm{Hom}_\Delta([m], [n]) \mid [n] \nsubseteq \alpha([m]) \cup \{k\}\}.$$

When $0 < k < n$, Λ_k^n is called an *inner horn*. When $k = 0$ or $k = n$, Λ_k^n is called an *outer horn*.

Geometrically, Λ_k^n is the simplicial subset of Δ^n obtained by removing its unique non-degenerate n-simplex and the unique non-degenerate $(n-1)$-simplex that does not contain the vertex k.

Example 6.1.26. We have $\Lambda_0^1 \simeq \Lambda_1^1 \simeq \Delta^0$, with the canonical inclusion maps $\Lambda_i^1 \subseteq \Delta^1$ given by

Example 6.1.27. The three horns Λ_i^2 ($0 \leq i \leq 2$) can be represented as follows:
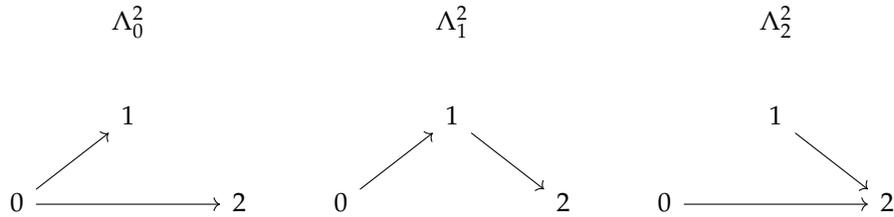
In particular, given a simplicial set X, the data of a map $\Lambda_1^2 \to X$ corresponds to a diagram of the form

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z$$

in X, while the data of maps $\Lambda_0^2 \to X$ and $\Lambda_2^2 \to X$ respectively correspond to diagrams in X of the following forms:

Fact 6.1.28 ([Lur18, Tag 0030])**.** *A simplicial set X is isomorphic to the nerve of a category if and only if for every $0 < k < n$ and every map $f : \Lambda_k^n \to X$, there exists a unique map $\sigma : \Delta^n \to X$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\ f\ } & X \\
\big\downarrow & \nearrow{\sigma} & \\
\Delta^n & &
\end{array}
$$

Moreover, X is isomorphic to the nerve of a groupoid if and only if the above condition is also satisfied when $k = 0$ and $k = n$.

In other words, a simplicial set "is" a category if and only if it admits unique lifts of maps from inner horns, and it "is" a groupoid if it also admits unique lifts of maps from outer horns. The singular complex associated to a topological space satisfies a similar, although somewhat different property:

Fact 6.1.29. *Let X be a topological space. Then for every $0 \le k \le n$ and every map $f : \Lambda_k^n \to S(X)$, there exists a (non-necessarily unique) map $\sigma : \Delta^n \to X$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\ f\ } & X \\
\big\downarrow & \nearrow{\sigma} & \\
\Delta^n & &
\end{array}
$$

Example 6.1.30. Let $\mathcal{C}$ be a category. A map $\Lambda_1^2 \to \mathrm{N}\left(\mathcal{C}\right)$ corresponds to a diagram

in $\mathcal{C}$, and a lifting $\Delta^2 \to \mathrm{N}\left(\mathcal{C}\right)$ of this map corresponds to a commutative triangle

in $\mathcal{C}$. That there always exists a unique lift means that a pair of composable morphisms always admits a unique composition in a category.

Example 6.1.31. Let X be a simplicial set that satisfies the hypothesis of fact 6.1.28. The existence of unique lifts for maps $\Lambda_1^2 \to X$ means we can define a composition law $- \circ -$ for edges in X. The triangles of example 6.1.15 witness that this composition is unital with respect to the identity edge id_x.

To see that it is associative, we can use a higher-dimensional horn-lifting condition: consider a sequence of edges in X as follows:

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \xrightarrow{\ h\ } t$$

Using the unique lifting conditions for the horn Λ_1^2, we can build a map $\Lambda_1^3 \to X$ represented as the following diagram:

$$(6.2)$$

The unique lifting condition for the horn Λ_1^3 gives us a 3-simplex $\tau \in X_3$ such that $d_1(\tau)$ fills the exterior triangle of diagram 6.2, so that we have a triangle

in X. But this triangle is also a lift for the map $\Lambda_1^2 \to X$ corresponding to the pair of morphisms $((g \circ f), h)$, so by unicity we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 6.1.32. Let X be a topological space, and $f : x \to y$ an edge in $S(X)$. By definition, this edge corresponds to a path in X, i.e. a continuous map $f : [0, 1] \to X$. Unfolding the definition of $S(X)$, we can also see that the identity edge $\mathrm{id}_x : x \to x$ corresponds to the constant path at $x \in X$. Now, the lifting condition for the outer horn Λ_0^2 implies in particular that the following diagram

of paths in X can be lifted to a triangle

i.e. an actual continuous map $\sigma : |\Delta^n| \to X$. This can be realized by taking $g(x) = f(-x)$, i.e. the reverse path of f. The triangle σ then just witnesses that the composition of a path with its reverse is *homotopic* to the constant path.

In more general simplicial sets, lifting outer horns is tantamount to finding inverses for edges, hence why this cannot be done in general for the nerve of a category, but it can be done for the nerve of a groupoid.

Remark 6.1.33. Given topological space X, the non-uniqueness of the horn-lifting comes from the fact there are many possible ways to compose paths in X. On the other hand, the fact that higher-dimensional horns can always be filled should be interpreted as saying that although there are many possible choices for composing paths, those any two choices of composition are always *homotopic*: this will motivate the notion of *homotopy* between edges in a simplicial set below (definition 6.2.6).

With these examples in mind, we are now ready to state the definition of the main objects of discourse of this chapter and the next.

6.2 ∞-categories and ∞-groupoids

Definition 6.2.1. A simplicial set X is said to be an ∞-*category* (or *quasicategory*) if for every $0 < k < n$ and morphism $h : \Lambda_k^n \to X$, there exists an n-simplex $\sigma \in X_n$ such that the following triangle commutes.

$$
\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\ h\ } & X \\
\downarrow & \nearrow_{\sigma} & \\
\Delta^n & &
\end{array}
\tag{6.3}
$$

If this condition also holds for $k = 0$ and $k = n$, we moreover say that X is an ∞-*groupoid* (or *Kan complex*).

We write qCat for the category of quasicategories and Kan for the category of Kan complexes.

Example 6.2.2. The singular complex $S(X)$ associated to a topological space X is an ∞-groupoid. The nerve $\mathrm{N}(\mathcal{C})$ of a category is an ∞-category.

Fact 6.2.3 ([Lur18, Tag 003S]). *If a simplicial set $\mathcal{C}$ is an ∞-category, so is $\mathcal{C}^{\mathrm{op}}$. Similarly, if $\mathcal{C}$ is an ∞-groupoid, so is $\mathcal{C}^{\mathrm{op}}$.*

Fact 6.2.4. *If $\mathcal{C}, \mathcal{D}$ are ∞-categories (respectively, ∞-groupoids), so is $\mathcal{C} \times \mathcal{D}$.*

Definition 6.2.5 ([Lur18, Tag 0041]). Let X be a simplicial set, $x, y, z \in X_0$, $f : x \to y$, $g : y \to z$ and $h : x \to z$ be edges in X. We say that h is *a composition* of f and g if there exists a triangle $\sigma \in X_2$ such that $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$. In other words, σ fills the following diagram in X:

$$
\begin{array}{ccc}
 & y & \\
{}^{f}\nearrow & {\sigma} & \searrow^{g} \\
x & \xrightarrow[h]{} & z
\end{array}
$$

We also say that σ *witnesses h as a composition of f and g*.

Definition 6.2.6. Let X be a simplicial set, $x, y \in X_0$, and $f, g : x \to y$ be edges from x to y in X. A *homotopy* from f to g in X is a triangle $\sigma \in X_2$ that witnesses g as a composition of f and id_x:

$$
\begin{array}{ccc}
 & y & \\
{}^{f}\nearrow & {\sigma} & \searrow^{\mathrm{id}_y} \\
x & \xrightarrow[g]{} & y
\end{array}
$$

If there is an homotopy from f to g, we say that f and g are *homotopic*, and write $f \sim g$.

In an ∞-category $\mathcal{C}$, the existence of lifts for the horn Λ_1^2 guarantees that compositions of morphisms always exist, but they have in general no reason to be unique. However, compositions are unique *up to homotopy*:

Fact 6.2.7 ([Lur18, Tag 0043])**.** *Let $\mathcal{C}$ be an ∞-category, and let $f : x \to y$, $g : y \to z$, and $h, h' : x \to z$ be morphisms in $\mathcal{C}$, and suppose that h is a composition of f and g in $\mathcal{C}$. Then h' is a composition of f and g if and only if $h \sim h'$.*

Moreover, the relation of homotopy is very well-behaved in ∞-categories:

Fact 6.2.8 ([Lur18, Tag 003Z])**.** *In an ∞-category $\mathcal{C}$, given objects $x, y \in \mathcal{C}_0$, the homotopy relation is an equivalence relation on the set of morphisms from x to y.*

Fact 6.2.9 ([Lur18, Tag 0040])**.** *Let $\mathcal{C}$ be an ∞-category, and $f, g : x \to y$ be morphisms in $\mathcal{C}$. Then $f \sim g$ in $\mathcal{C}$ if and only if $f \sim g$ when seen as morphisms in $\mathcal{C}^{\mathrm{op}}$.*

Fact 6.2.10 ([Lur18, Tag 0048])**.** *In an ∞-category, the homotopy relation is compatible with compositions: if $f \sim f'$ and $g \sim g'$, h is a composition of g and f and h' is a composition of g' and f', then $h \sim h'$.*

Fact 6.2.11 ([Lur18, Tag 004B])**.** *In an ∞-category, the homotopy relation is associative and unital with respect to all compositions and with units given by the identity morphisms id_x.*

Relying on these facts, we can define the homotopy category of an ∞-category:

Definition 6.2.12. Let $\mathcal{C}$ be an ∞-category. Its *homotopy category* is the category $\mathrm{ho}\,\mathcal{C}$ whose objects are the objects of $\mathcal{C}$, i.e. the elements of $\mathcal{C}_0$, whose morphisms are the homotopy classes of morphisms in $\mathcal{C}$, and where the composition is induced by the existence and unicity up to homotopy of compositions in $\mathcal{C}$.

Definition 6.2.13. Let $\mathcal{C}$ be an ∞-category, and let $f : x \to y$ be a morphism in $\mathcal{C}$. We say that f is an *isomorphism* in $\mathcal{C}$ if the induced morphism $[f] : x \to y$ in $\mathrm{ho}\,\mathcal{C}$ is an isomorphism in $\mathrm{ho}\,\mathcal{C}$.

In other words, a morphism is an isomorphism if it admits both a left and right inverse up to homotopy.

Remark 6.2.14. Just like for the definition of isomorphisms in wild categories, we only need low-dimensional data to be able to define isomorphisms (morphisms and homotopies).

Fact 6.2.15. [*Lur18, Tag 019D*] *An ∞-category is an ∞-groupoid if and only if all its morphisms are isomorphisms.*

Remark 6.2.16. Fact 6.2.15 was first proven by Joyal in [Joy02]. Even though its statement seems simple enough, it was actually deceptively difficult to prove: Joyal himself explained in an email to Dmitri Pavlov that he failed to prove this result for about 15 years before finally being able to prove it by extending usual notions from category theory to the setting of quasicategories[Pav22]. Historically, the term ∞-groupoid is reserved for ∞-categories whose morphisms are all invertible, so this statement can also be read as

<p align="center">"∞-groupoids coincide with Kan complexes."</p>

We can actually state something a bit more general:

Definition 6.2.17 ([Lur18, Tag 01CZ])**.** Let $\mathcal{C}$ be an ∞-category. The *core* of $\mathcal{C}$, denoted $\mathcal{C}^{\simeq}$, is the simplicial subset of $\mathcal{C}$ on the simplices whose underlying edges are all isomorphisms in $\mathcal{C}$.

Fact 6.2.18 ([Lur18, Tag 01H1]). *The simplicial set $\mathcal{C}^{\simeq}$ is an ∞-groupoid.*

We now turn our attention to functors of ∞-categories.

Definition 6.2.19. A morphism of simplicial sets $f : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories is called a *functor*.

Definition 6.2.20 ([Lur18, Tag 005Z]). The category sSet is cartesian closed (by virtue of being a presheaf category). Given simplicial sets X and Y, we write $\mathrm{Fun}\,(X, Y)$ for the internal hom from X to Y in sSet. Explicitly this is given by:

$$\mathrm{Fun}\,(X, Y)_n := \mathrm{Hom}_{\mathrm{sSet}}\,(\Delta^n \times X, Y).$$

Remark 6.2.21. The bijection

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(X^{\mathrm{op}}, Y^{\mathrm{op}})$$
$$F \mapsto F^{\mathrm{op}}$$

of definition 6.1.18 extends to an isomorphism of simplicial sets

$$\mathrm{Fun}\,(X, Y) \simeq \mathrm{Fun}\,(X^{\mathrm{op}}, Y^{\mathrm{op}})^{\mathrm{op}}.$$

Fact 6.2.22 ([Lur18, Tag 0066]). *Let X be a simplicial set and $\mathcal{C}$ an ∞-category. Then $\mathrm{Fun}\,(X, \mathcal{C})$ is an ∞-category. If moreover $\mathcal{C}$ is an ∞-groupoid, so is $\mathrm{Fun}\,(X, \mathcal{C})$.*

Fact 6.2.23. *Given simplicial sets X and Y, the internal hom $[X, Y]$ is a Kan complex if Y is a Kan complex, and it is a quasicategory if Y is a quasicategory.*

Definition 6.2.24. Let X be a simplicial set, $\mathcal{C}$ an ∞-category, and let $F, G : X \to \mathcal{C}$ be morphisms of simplicial sets. A morphism $\alpha : F \to G$ in $\mathrm{Fun}\,(X, \mathcal{C})$ is called a *natural transformation*, and we write $\alpha : F \Rightarrow G$. It said to be a *natural isomorphism* if it is an isomorphism in $\mathrm{Fun}\,(X, \mathcal{C})$, in which case we may also write $\alpha : F \simeq G$.

Unfolding the definitions, a natural transformation $\alpha : F \Rightarrow G$ is a morphism of simplicial sets

$$\alpha : \Delta^1 \times X \to \mathcal{C}$$

such that $\alpha|_{\{0\} \times X} = F$ and $\alpha|_{\{1\} \times X} = G$. In particular, for every $x \in X_0$, the composition

$$\Delta^1 \xrightarrow{\simeq} \Delta^1 \times \Delta^0 \xrightarrow{\mathrm{id}_{\Delta^1} \times x} \Delta^1 \times X \xrightarrow{\alpha} \mathcal{C}$$

determines a morphisms $\alpha_x : F(x) \to G(x)$ in $\mathcal{C}$. One dimension higher, any morphism $f : x \to y$ in X yields a map

$$\Delta^1 \times \Delta^1 \xrightarrow{\mathrm{id}_{\Delta^1} \times f} \Delta^1 \times X \xrightarrow{\alpha} \mathcal{C}$$

which may be represented as a commutative diagram

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
\alpha_x \downarrow & \searrow^{\alpha_f} & \downarrow \alpha_y \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
$$

in $\mathcal{C}$ — called the *naturality square* for α — witnessing the naturality of α in the classical sense, up to homotopy.

Fact 6.2.25 ([Lur18, Tag 01DK]). *A natural transformation* $\alpha : F \Rightarrow G : X \to \mathcal{C}$ *is a natural isomorphism if and only if for every* $x \in X_0$, *the morphism* $\alpha_x : F(x) \to G(x)$ *in* $\mathcal{C}$ *is an isomorphism in* $\mathcal{C}$.

Remark 6.2.26. Fact 6.2.25 is straightforward enough to prove in the setting of categories, since there naturality is just a condition on a family of morphisms. In the ∞-categorical setting however, the "naturality" of a family of morphisms corresponds to the data of additional structure in every dimension, so trying to prove fact 6.2.25 by directly constructing an inverse natural transformation is essentially impossible. The actual proof in [Lur18, Tag 01DF] relies on a careful sequence of lemmas on the combinatorics of simplicial sets.

Fact 6.2.27 ([Lur18, Tag 0062]). *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. There is an isomorphism of* ∞-*categories*

$$\text{Fun}\left(\text{N}\,\mathcal{C}, \text{N}\,\mathcal{D}\right), \text{N}\left(\text{Fun}(\mathcal{C}, \mathcal{D})\right)$$

where $\text{Fun}(\mathcal{C}, \mathcal{D})$ *is the usual category of functors from* $\mathcal{C}$ *to* $\mathcal{D}$.

6.3 Equivalences of ∞-categories

The definition of equivalences of categories can be easily adapted to the ∞-categorical setting:

Definition 6.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that F is an *equivalence of* ∞-*categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $G \circ F \simeq \text{id}_{\mathcal{C}}$ and $F \circ G \simeq \text{id}_{\mathcal{D}}$, in which case we write $F : \mathcal{C} \simeq \mathcal{D}$. If there is an equivalence $F : \mathcal{C} \to \mathcal{D}$, we say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*.

When $\mathcal{C}$ and $\mathcal{D}$ are ∞-groupoids, we also say that F is a *homotopy equivalence*, and that $\mathcal{C}$ and $\mathcal{D}$ are *homotopy equivalent*.

This notion of equivalence gives rise to two notions of equivalence for more general simplicial sets:

Definition 6.3.2. Let X, Y be simplicial sets, and $f : X \to Y$ be a morphism of simplicial sets.

- $f : X \to Y$ is said to be a *weak homotopy equivalence* if for every ∞-groupoid Z, the map

$$(f \circ -) : \text{Fun}\left(Y, Z\right) \to \text{Fun}\left(X, Z\right)$$

  is a homotopy equivalence of ∞-groupoids.

- $f : X \to Y$ is said to be a *weak categorical equivalence* if for every ∞-category $\mathcal{C}$, the map

$$(f \circ -) : \text{Fun}\left(Y, Z\right) \to \text{Fun}\left(X, Z\right)$$

  is an equivalence of ∞-categories.

Note that in both cases, the simplicial sets $\text{Fun}\left(X, Z\right)$ and $\text{Fun}\left(Y, Z\right)$ are respectively ∞-groupoids and ∞-categories by virtue of fact 6.2.22.

From the point of view of the theory of model categories, an arbitrary simplicial set X can be thought of as presenting both an ∞-groupoid and an ∞-category, depending on the class of weak equivalences chosen on sSet. In particular, if $f : \mathcal{C} \to X$ is a weak homotopy equivalence from an ∞-category to an ∞-groupoid, then X can be thought of as the ∞-groupoid obtained by formally inverting every morphism in $\mathcal{C}$.

Remark 6.3.3 (The homotopy hypothesis for Kan complexes). There is a model category structure on sSet, called the *Kan-Quillen model structure*, in which the fibrant objects are the Kan complexes, and the weak equivalences are the weak homotopy equivalences. This model category is Quillen equivalent to the category of topological spaces with the Quillen model structure, in which the weak equivalences are the weak homotopy equivalences of topological spaces: this is the content of the *homotopy hypothesis* for Kan complexes, the formal statement that Kan complexes are models for homotopy types.

Remark 6.3.4. A functor $f : X \to Y$ between ∞-groupoids is a homotopy equivalence if and only if it is a weak homotopy equivalence. Similarly, a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is an equivalence if and only if it is a weak categorical equivalence.

Definition 6.3.5. An ∞-groupoid is said to be *contractible* if it is homotopy equivalent to Δ^0. A simplicial set is said to be *weakly contractible* if it is weakly homotopy equivalent to Δ^0.

In category theory, equivalences can also be characterized as the *essentially surjective and fully faithful functors*. To state a similar result for ∞-categories, we first need to talk about hom spaces.

Definition 6.3.6. Let X be a simplicial set, and $x, y \in X_0$. The *hom space* $\mathrm{Hom}_X(x, y)$ (or simply $\mathrm{Hom}(x, y)$ for short) is defined as the following pullback in the category sSet:

$$
\begin{array}{ccc}
\mathrm{Hom}_X(x,y) & \longrightarrow & \mathrm{Fun}\left(\Delta^1, X\right) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (f:z\to t)\mapsto[z,t]} \\
\Delta^0 & \xrightarrow[{[x,y]}]{} & \mathrm{Fun}\left(\Delta^0 \sqcup \Delta^0, X\right)
\end{array}
$$

The terminology "hom space" is due to the following fact (together with the fact that ∞-groupoids are typically thought of as spaces):

Fact 6.3.7 ([Lur18, Tag 01JC]). *If $\mathcal{C}$ be an ∞-category and x, y are objects in $\mathcal{C}$, then $\mathrm{Hom}_\mathcal{C}(x, y)$ is an ∞-groupoid. Moreover, for every pair of morphisms $f, g : x \to y$ in $\mathcal{C}$, f and g are homotopic if and only if there is a morphism $u : f \to g$ in $\mathrm{Hom}_\mathcal{C}(x, y)$ ([Lur18, Tag 01JD]).*

Remark 6.3.8. Fact 6.3.7 justifies thinking of ∞-categories as "categories internal to homotopy theory": since ∞-groupoids are representations for homotopy types (remark 6.3.3), an ∞-category $\mathcal{C}$ admits:

- an *underlying* homotopy type $\mathcal{C}^{\simeq}$,

- for every objects x, y, a *homotopy type* $\mathrm{Hom}_\mathcal{C}(x, y)$.

Moreover, it can be shown that the mere existence of compositions of morphisms in $\mathcal{C}$ can be enhanced to a composition functor:

$$
\mathrm{Hom}_\mathcal{C}(y, z) \times \mathrm{Hom}_\mathcal{C}(x, y) \to \mathrm{Hom}_\mathcal{C}(x, z)
$$

in a way that is associative and unital up to natural isomorphism ([Lur18, Tag 01PQ],[Lur18, Tag 01PF]): in that sense, ∞-categories can be thought of as *wild categories* with additional structure.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and $x, y \in \mathcal{C}_0$. The universal property of the pullback definition of $\mathrm{Hom}_\mathcal{D}(F(x), F(y))$ allows us to build a canonical functor

$$
\mathrm{Hom}_\mathcal{C}(x, y) \to \mathrm{Hom}_\mathcal{D}(F(x), F(y)).
$$

Definition 6.3.9 ([Lur18, Tag 01JH])**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if for every $x, y \in \mathcal{C}_0$, the induced functor

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$$

is an equivalence.

Definition 6.3.10. A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every object $y \in \mathcal{D}_0$, there exists an object $x \in \mathcal{C}_0$ and an isomorphism $f : F(x) \simeq y$ in $\mathcal{D}$.

Fact 6.3.11 ([Lur18, Tag 01JX])**.** *A functor $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective.*

From the point of view of homotopy theory, hom spaces can also be used to define loop spaces:

Definition 6.3.12. Let X be an ∞-groupoid, and $x \in X_0$. The *loop space* of X at x, denoted $\Omega(X, x)$, is the ∞-groupoid $\mathrm{Hom}_X(x, x)$. Note that $\Omega(X, x)$ is canonically pointed at $\mathrm{id}_x \in \Omega(X, x)$, so we can iterate the loop space construction and write $\Omega^n(X, x)$.

And we can in turn use loop spaces to define homotopy groups:

Definition 6.3.13. Let X be an ∞-groupoid. We define $\pi_0(X)$ to be the quotient of X_0 by the relation of "being isomorphic".

Given $x \in X_0$ and $n \geq 1$, the n-th *homotopy group* of X at x is the group

$$\pi_n(X, x) := \pi_0(\Omega^n(X, x)),$$

where the group law is given by the composition of morphisms in $\Omega^{n-1}(X, x)$.

Note that we do not use exactly the same definition than in [Lur18, Tag 00V2] for brevity of the exposition, but the resulting homotopy groups are the same. We can recover important results from homotopy theory in the formalism of ∞-groupoids:

Fact 6.3.14. *Let (X, x) be a pointed ∞-groupoid, then the group $\pi_n(X, x)$ is abelian for $n \geq 2$.*

Fact 6.3.15 (Whitehead's theorem for Kan complexes [Lur18, Tag 00VU])**.** *Let $F : X \to Y$ be a functor between ∞-groupoids. Then F is a homotopy equivalence if and only for every $x \in X_0$ and $n \geq 0$, the map*

$$\pi_n(F) : \pi_n(X, x) \to \pi_n(Y, F(x))$$

is an isomorphism of groups (or a bijection when $n = 0$).

As an immediate corollary:

Fact 6.3.16. *An ∞-groupoid X is contractible if and only if all its homotopy groups are trivial.*

6.4 Fibrations and the straightening/unstraightening equivalence

6.4.1 Fibrations of simplicial sets

The isofibrations of section 2.6.3 and the Grothendieck (op)fibrations of section 2.6.4 have counterparts in the realm of ∞-categories — respectively called isofibrations and (co)cartesian fibrations — in which they play an even more prominent role, notably in the definition of hom functors and (symmetric) monoidal structures. Fibrations are of such great importance to the theory of
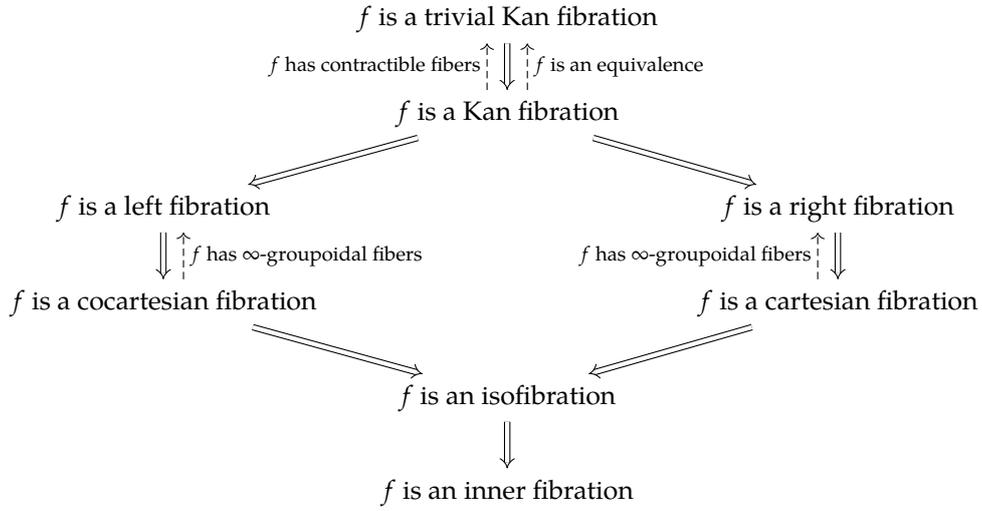
f is a trivial Kan fibration

f has contractible fibers $\quad$ f is an equivalence

f is a Kan fibration

f is a left fibration $\qquad\qquad\qquad\qquad$ f is a right fibration

f has ∞-groupoidal fibers $\qquad\qquad$ f has ∞-groupoidal fibers

f is a cocartesian fibration $\qquad\qquad$ f is a cartesian fibration

f is an isofibration

f is an inner fibration

Figure 6.2: Conditions on a functor $f : \mathcal{C} \to \mathcal{D}$ ∞-categories and their different implications, with conditional reverse implications written in dotted arrows.

∞-categories that many other notions of fibrations are often considered: they are summed up in fig. 6.2.

Most of these notions of fibrations are defined using lifting properties, so we first recall their general terminology:

Definition 6.4.1. In a category $\mathcal{C}$, a morphism $f : x \to y$ has the *right lifting property* (RLP) with respect to $i : a \to b$ if for every commutative square

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & x \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
b & \xrightarrow{\ k\ } & y
\end{array}
$$

in $\mathcal{C}$, there exists a morphism $g : b \to y$ such that $f \circ g = k$ and $g \circ i = h$, i.e. the following two triangles commute:

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & x \\
{\scriptstyle i}\downarrow & \nearrow{\scriptstyle g} & \downarrow{\scriptstyle f} \\
b & \xrightarrow{\ k\ } & y
\end{array}
$$

Definition 6.4.2. A map $f : X \to Y$ of simplicial sets is said to be:

1. a *trivial Kan fibration* if it has the RLP with respect to $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$,[1]

2. a *Kan fibration* if it has the RLP with respect to $\Lambda_k^n \hookrightarrow \Delta^n$ for all $0 \leq k \leq n$,

---

[1]Here, $\partial\Delta^n$ denotes the simplicial subset of Δ^n obtained by removing only its unique non-degenerate n-dimensional simplex.

3. a *left fibration* if it has the RLP with respect to $\Lambda_k^n \hookrightarrow \Delta^n$ for all $0 \le k < n$,

4. a *right fibration* if it has the RLP with respect to $\Lambda_k^n \hookrightarrow \Delta^n$ for all $0 < k \le n$,

5. an *inner fibration* if it has the RLP with respect to $\Lambda_k^n \hookrightarrow \Delta^n$ for all $0 < k < n$

Example 6.4.3. By definition, a simplicial set X is an ∞-groupoid if and only if the unique map $X \to \Delta^0$ is a Kan fibration, and it is an ∞-category if and only if that map is an inner fibration.

The definition of isofibrations is similar to definition 2.6.9:

Definition 6.4.4 ([Lur18, Tag 01ES])**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is said to be an *isofibration* if it is an inner fibration and for every $x \in \mathcal{C}_0$, $y \in \mathcal{D}_0$ and isomorphism $f : y \to F(x)$ in $\mathcal{D}$, there exists an isomorphism $\bar{f} : \bar{y} \to x$ in $\mathcal{C}$ such that $F(\bar{f}) = f$.

As we mentioned in section 2.6.3, a functor $F : \mathcal{C} \to \mathcal{D}$ between categories being an isofibration means that it respects the principle of equivalence: two isomorphic objects in $\mathcal{D}$ will have equivalent fibers. When $F : \mathcal{C} \to \mathcal{D}$, the same intuition still holds, and the condition of being an inner fibration is added to prevent another way in which the principle of equivalence can fail: if $F : \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories, $f : x \to y$, $g : y \to z$ are morphisms in $\mathcal{C}$ and $u : F(x) \to F(z)$ is a composition of $F(f)$ and $F(g)$ in $\mathcal{D}$, then there is no guarantee that there exists a composition h of f and g in $\mathcal{C}$ such that $F(h) = u$, even though there will always be a homotopy $F(h) \sim u$. Said differently, the condition of F being an inner fibration in addition to admitting lifting for isomorphisms guarantees that the induced functors $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ will also admit lifting for isomorphisms (and even be isofibrations [Lur18, Tag 01P8]).

Remark 6.4.5. The isofibrations between ∞-categories are a particular case of the more general notion of *categorical fibration* between simplicial sets. The categorical fibrations are the class of fibrations for the so-called *Joyal model structure* on sSet, whose weak equivalences are the weak categorical equivalences. Giving a simple description for categorical fibrations is quite hard in general, but when $f : X \to \mathcal{D}$ is a morphism of simplicial sets such that $\mathcal{D}$ is an ∞-category, then f is a categorical fibration if and only if it is an isofibration [Lur09, Cor 2.4.6.5]. Hence, when reading [Lur09] or [Lur17], one may safely replace the term "categorical fibration" with "isofibration" in most statements.

Inner fibrations can be used to define subcategories of ∞-categories:

Definition 6.4.6 ([Lur18, Tag 01CF])**.** Let $\mathcal{C}$ be an ∞-category. A *subcategory* of $\mathcal{C}$ is a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ such that the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an inner fibration.

We use terminology of *subcategory* rather than "sub-∞-category" as to not clutter notation.

Definition 6.4.7. Let $\mathcal{C}$ be an ∞-category, and $X \subseteq \mathcal{C}_0$ a set of objects in $\mathcal{C}$. The *full subcategory of $\mathcal{C}$ spanned by X* is the subcategory $\mathcal{C}'$ of $\mathcal{C}$ defined by

$$\mathcal{C}'_n := \{\sigma \in \mathcal{C}_n \mid \text{all the vertices of } \sigma \text{ belong to } X\}.$$

Definition 6.4.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The *essential image* of F is the full subcategory of $\mathcal{D}$ spanned by the objects isomorphic to one of the form $F(x)$ for $x \in \mathcal{C}_0$.

Remark 6.4.9. By definition, every functor is essentially surjective onto its essential image.

Finally, we need to define (co)cartesian fibrations. In order to do that, we need to find an ∞-categorical counterpart to the definition 2.6.13 of F-cartesian morphisms, which relied on a *unique lifting condition*. As we have seen repeatedly through chapters 4 and 5 in the setting of homotopy type theory, the homotopically meaningful way of stating that some data is *unique* is to say that the space of such data is *contractible*. When the data in question is specified as the fiber of a map f, we can equivalently require the map f to be a homotopy equivalence:

Definition 6.4.10 ([Lur09, Prop 2.4.4.3])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories. A morphism $f : x \to y$ in $\mathcal{C}$ is said to be *F-cartesian* if for every object $z \in \mathcal{C}_0$, the following commutative square is a *homotopy pullback square*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x,z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}}(F(x),F(y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F(x),F(z))
\end{array}
$$

i.e. the canonical map

$$
\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x),F(y)) \times_{\mathrm{Hom}_{\mathcal{D}}(F(x),F(z))} \mathrm{Hom}_{\mathcal{C}}(x,z)
$$

is a homotopy equivalence of ∞-groupoids.

Dually, f is said to be *F-cocartesian* if it is F^{op}-cartesian (where $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$).

We stated the definition only in the case when F is an inner fibration to simplify the exposition, see [Lur18, Tag 01T4] for a more general definition.

Definition 6.4.11 ([Lur18, Tag 01UA])**.** An inner fibration $F : \mathcal{C} \to \mathcal{D}$ is said to be a *cocartesian fibration* (resp. *cartesian fibration*) if for every morphism $f : x \to y$ in $\mathcal{D}$, there exists an F-cocartesian morphism (resp. F-cartesian morphism) $\tilde{f} : \tilde{x} \to \tilde{y}$ in $\mathcal{C}$ such that $F(\tilde{f}) = f$. In that case $\tilde{f}$ is said to be a *cocartesian lift* (resp. *cartesian lift*) of f with respect to F.

To understand the meaning of all those different kinds of fibrations, we need to take a look at the behaviour of their *fibers*.

Definition 6.4.12. Let $f : X \to Y$ be a map of simplicial sets, and y be a vertex of Y. The *fiber* $f^{-1}(y)$ of f at y is the simplicial set defined as

$$
f^{-1}(y) := \{y\} \times_Y X,
$$

i.e. as the pullback

$$
\begin{array}{ccc}
f^{-1}(y) & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
\Delta^0 & \xrightarrow{\ y\ } & Y
\end{array}
$$

Fact 6.4.13. *Let $F : X \to Y$ be a morphism of simplicial sets. We have the following:*

If F is a	then its fibers are	and every $f : x \to y$ in Y induces a
trivial Kan fibration	contractible ∞-groupoids	homotopy equivalence $f_! : F^{-1}(x) \simeq F^{-1}(y)$
Kan fibration	∞-groupoids	homotopy equivalence $f_! : F^{-1}(x) \simeq F^{-1}(y)$
left fibration	∞-groupoids	functor $f_! : F^{-1}(x) \to F^{-1}(y)$
right fibration	∞-groupoids	functor $f^* : F^{-1}(y) \to F^{-1}(x)$
cocartesian fibration	∞-categories	functor $f_! : F^{-1}(x) \to F^{-1}(y)$
cartesian fibration	∞-categories	functor $f^* : F^{-1}(y) \to F^{-1}(x)$
isofibration or an inner fibration	∞-categories	/

Moreover, if F is a trivial Kan fibration and Y is an ∞-category, then F is an equivalence of ∞-categories.

We can say even more about the functors induced between the fibers of F: they are compatible with compositions of morphisms in Y, in the sense that $g \circ f_! \simeq g_! \circ f_!$. Going even further, the assignment $f \mapsto f_!$ defines a functor to the ∞-category of ∞-categories: this is the statement of the straightening/unstraightening equivalence, the indexed/fibred correspondence for ∞-categories (fact 6.4.27).

6.4.2 The ∞-categories $\mathcal{S}$ and $\infty\,\mathrm{Cat}$

Before defining the ∞-category of ∞-categories, we briefly need to address size issues, as there is no such a thing a *set of all ∞-categories*. We avoid this foundational problem by working with *Grothendieck universes*:

Definition 6.4.14. A *Grothendieck universe* (or *universe* for short) is a set $\mathcal{U}$ which is closed under all the constructions of set theory:

1. $\varnothing \in \mathcal{U}$,

2. if $A, B \in \mathcal{U}$, $\{A, B\} \in \mathcal{U}$,

3. if $X \in \mathcal{U}$, $\cup_{Y \in X} Y \in \mathcal{U}$,

4. if $X \in \mathcal{U}$, $P(X) \in \mathcal{U}$,

5. if $X \in Y \in \mathcal{U}$, $X \in \mathcal{U}$.

Equivalently, a Grothendieck universe is the set of sets of *rank* less than κ for κ some *strongly inaccessible cardinal*.

Axiom 6.4.15 (Tarski-Grothendieck set theory). For every set X, we assume there exists a universe $\mathcal{U}$ such that $X \in \mathcal{U}$. More specifically, we fix three universes such that $\mathbb{N} \in \mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2$.

Remark 6.4.16. Grothendieck universes play a similar role in set theory that universe play in type theory, and the closure assumptions of definition 6.4.14 can be read as a set-theoretic flavour of the closure of type theoretical universes under dependent sums and dependent product types.

Definition 6.4.17. A set X is said to be $\mathcal{U}$-*small* if $X \in \mathcal{U}$. A simplicial set X is said to be $\mathcal{U}$-small if for every $n \in \mathbb{N}$, X_n is $\mathcal{U}$-small. An ∞-category $\mathcal{C}$ is said to be *essentially $\mathcal{U}$-small* if it is equivalent to a $\mathcal{U}$-small ∞-category.

In our setting, $\mathcal{U}_0$-small things will be called *small* for short, $\mathcal{U}_1$-small things will be called *large*, and $\mathcal{U}_2$-small things will be called *very large*.

Remark 6.4.18. More generally, we will talk of $\mathcal{C}$ being essentially κ-small for cardinals $\kappa > \aleph_0$.

Definition 6.4.19. An ∞-category $\mathcal{C}$ is said to be *locally $\mathcal{U}$-small* if for every $x, y \in \mathcal{C}_0$, the ∞-groupoid $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is essentially $\mathcal{U}$-small.

Fact 6.4.20 ([Lur18, Tag 03TW]). *An ∞-category $\mathcal{C}$ is essentially $\mathcal{U}$-small if and only if it is locally $\mathcal{U}$-small and the set $\pi_0(\mathcal{C}^{\simeq})$ of isomorphism classes of objects in $\mathcal{C}$ is isomorphic to a $\mathcal{U}$-small set.*

We can finally define the ∞-category of ∞-categories using a variant of the nerve construction for Kan -enriched categories:

Definition 6.4.21 ([Lur18, Tag 00KS])**.** Similarly to the nerve $\mathrm{N}\,\mathcal{C}$ of a category $\mathcal{C}$ (example 6.1.21), one can define the *homotopy coherent nerve* $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})$ of a *category (strictly) enriched in Kan complexes* $\mathcal{C}$ (see [Lur18, Tag 00KM] for a more complete treatment).

Fact 6.4.22 ([Lur18, Tag 01LF])**.** *Let $\mathcal{C}$ be a category enriched in Kan complexes. The objects of $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})$ are the objects of $\mathcal{C}$, and for every pair of objects x, y in $\mathcal{C}$ there is a canonical homotopy equivalence*

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \simeq \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{C})}(x, y).$$

Definition 6.4.23. The category Kan of small Kan complexes is cartesian closed (fact 6.2.22), so it can be enriched over itself by setting

$$\mathrm{Hom}_{\mathrm{Kan}}(X, Y) := \mathrm{Fun}(X, Y).$$

The *∞-category of ∞-groupoids*, denoted $\mathcal{S}$ (for *spaces*) is the homotopy coherent nerve of Kan with this canonical enrichment:
$$\mathcal{S} := \mathrm{N}^{\mathrm{hc}}(\mathrm{Kan}).$$

Definition 6.4.24. The category qCat of small quasicategories is cartesian closed (fact 6.2.22), so it can be enriched over Kan by setting

$$\mathrm{Hom}_{\mathrm{qCat}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}.$$

The *∞-category of ∞-categories*, denoted $\infty\,\mathrm{Cat}$, is the homotopy coherent nerve of qCat with this canonical enrichment:
$$\infty\,\mathrm{Cat} := \mathrm{N}^{\mathrm{hc}}(\mathrm{qCat}).$$

Remark 6.4.25. If $\mathcal{C}$ is a sufficiently well-behaved category enriched in Kan complexes (in the strict, 1-categorical sense of enrichment), then canonical morphism $\mathrm{N}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}(\mathcal{C})$ exhibits $\mathrm{N}^{\mathrm{hc}}(\mathcal{C})$ as an ∞-categorical localization of $\mathcal{C}$ at the class of weak equivalences given by morphisms that are invertible up to homotopy [htt]. This applies to both the above examples, so that $\infty\,\mathrm{Cat}$ and $\mathcal{S}$ can respectively be realized as localizations of qCat and Kan.

The last ingredient we need for fact 6.4.27 is the definition of *slice ∞-categories*:

Definition 6.4.26. Given an ∞-category $\mathcal{C}$ and an object $x \in \mathcal{C}_0$, the *slice ∞-category* $\mathcal{C}_{/x}$ is defined as the following pullback in sSet:

$$
\begin{array}{ccc}
\mathcal{C}_{/x} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (f:y\to z)\mapsto z} \\
\Delta^0 & \xrightarrow{\ x\ } & \mathcal{C}
\end{array}
$$

Dually, the *coslice* $\mathcal{C}_{x/}$ is defined to be $\mathcal{C}_{x/} := ((\mathcal{C}^{\mathrm{op}})_{/x})^{\mathrm{op}}$.

Unfolding the definition, the objects in $\mathcal{C}_{/x}$ are the morphisms in $\mathcal{C}$ with target x, and the morphisms in $\mathcal{C}_{/x}$ from $f : y \to x$ to $g : z \to x$ correspond to commutative diagrams

$$
\begin{array}{ccc}
y & \xrightarrow{\ h\ } & z \\
{\scriptstyle f}\downarrow & \searrow^{h'} & \downarrow{\scriptstyle g} \\
x & \xrightarrow[\ \mathrm{id}_x\ ]{} & x
\end{array}
$$

in $\mathcal{C}$, which up to homotopy is equivalent to the data of a commutative triangle

$$
\begin{array}{ccc}
y & \xrightarrow{\quad h \quad} & z \\
 & {}_{f}\searrow \quad \swarrow{}_{g} & \\
 & x &
\end{array}
$$

in $\mathcal{C}$, so we can think of $\mathcal{C}_{/x}$ as an ∞-categorical variant of the classical slice category construction. We can now state the indexed/fibred correspondence for ∞-categories:

Fact 6.4.27 (Straightening/unstraightening equivalence, [Lur09, Thm 3.2.0.1]). *Let $\mathcal{C}$ be an ∞-category.*

- *We write $(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{left fib.}}$ (resp. $(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{right fib.}}$) for the full subcategory of $\infty\,\mathrm{Cat}_{/\mathcal{C}}$ whose objects are the left fibrations (resp. right fibrations).*

- *We write $(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{cocart. fib.}}$ (resp. $(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{cocart. fib.}}$) for the subcategory of $\infty\,\mathrm{Cat}_{/\mathcal{C}}$ whose objects are the cocartesian fibrations (resp. cartesian fibrations) and whose morphisms are the functors that map cocartesian morphisms to cocartesian morphisms (resp. cartesian morphisms to cartesian morphisms).*

There are canonical equivalences:

$$
(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{left fib.}} \simeq \mathrm{Fun}\,(\mathcal{C}, \mathcal{S})
$$
$$
(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{right fib.}} \simeq \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathcal{S})
$$
$$
(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{cocart. fib.}} \simeq \mathrm{Fun}\,(\mathcal{C}, \infty\,\mathrm{Cat}\,)
$$
$$
(\infty\,\mathrm{Cat}_{/\mathcal{C}})^{\text{cart. fib.}} \simeq \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \infty\,\mathrm{Cat}\,)
$$

and these equivalences are moreover natural in $\mathcal{C}$ [GHN17, Cor A.32].

For a more self-contained description of the functor $f_! : F^{-1}(x) \to F^{-1}(y)$ associated to a morphism $f : x \to y$ in $\mathcal{D}$ given a cocartesian fibration $F : \mathcal{C} \to \mathcal{D}$, see [Lur18, Tag 019A].

Definition 6.4.28 ([Lur18, Tag 019N]). Let $F : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration and $f : x \to y$ in $\mathcal{D}$. The induced functor $f_! : F^{-1}(x) \to F^{-1}(y)$ is called The functor $\mathcal{D} \to \infty\,\mathrm{Cat}$ induced by the equivalence of fact 6.4.27 is called the *covariant transport functor* associated to f. If F is a cartesian fibration instead, the induced functor $f^* : F^{-1}(y) \to F^{-1}(x)$ is called the *contravariant transport functor* associated to f.

Remark 6.4.29. Thinking of ∞-groupoids as represented by types in homotopy type theory, a functor $F : X \to \mathcal{S}$ with X an ∞-groupoid should be thought of as a type family $F : X \to \mathcal{U}$, where $\mathcal{U}$ is a universe of types. When X is an ∞-groupoid, left fibrations, right fibrations and Kan fibrations over X coincide, and the Kan fibration

$$
p : \widetilde{X} \to X
$$

associated to the functor F corresponds in homotopy type theory to the projection from the dependent sum type

$$
\sum_{x:X} F(x) \to X.
$$

Under this correspondence, the covariant transport functor $p_!$ associated to an isomorphism $p : x \to y$ in X corresponds to the transport over p in the sense of homotopy type theory. More generally, fact 6.4.27 is the ∞-categorical version of fact 4.9.3.

Finally, the dependent product type $\prod_{x:X} F(x)$ corresponds in ∞-categorical terms to the ∞-groupoid of *sections* of $p : \widetilde{X} \to X$:

$$\prod_{x:X} F(x) \text{ "}\simeq\text{" } \mathrm{Fun}_{/X}\,(X, \widetilde{X})$$

with the notation of definition 6.4.30.

Because of the importance of $\infty\,\mathrm{Cat}_{/\mathcal{C}}$, we will often need in what follows to consider functors over a base ∞-category:

Definition 6.4.30 ([Lur18, Tag 01AB]). Let $f : X \to S$ and $g : Y \to S$ be morphisms of simplicial sets. We write $\mathrm{Fun}_{/S}\,(X, Y)$ for the simplicial subset of $\mathrm{Fun}\,(X, Y)$ obtained as the fiber of the postcomposition map

$$(g \circ -) : \mathrm{Fun}\,(X, Y) \to \mathrm{Fun}\,(X, S)$$

over the vertex $f \in \mathrm{Fun}\,(X, S)_0$.

Concretely, the objects of $\mathrm{Fun}_{/S}\,(X, Y)$ can be identified with the morphisms $h : X \to Y$ such that the following triangle commutes in sSet:

When X, Y and S are ∞-categories, we moreover have a canonical homotopy equivalence

$$\mathrm{Hom}_{\infty\,\mathrm{Cat}_{/S}}(X, Y) \simeq \mathrm{Fun}_{/S}\,(X, Y)^{\simeq}.$$

6.5 Hom functors and the Yoneda lemma

We are now going to apply fact 6.4.27 to enhance the hom spaces $\mathrm{Hom}_{\mathcal{C}}(x, y) \in \mathcal{S}$ to a *hom functor*

$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}.$$

If $\mathcal{C}$ is an ordinary category, the sets $\mathrm{Hom}_{\mathcal{C}}(x, y)$ act functorially on x and y via pre-composition and post-composition of morphisms. The discrete fibration over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ induced by this hom functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ is called the *twisted arrow category* of $\mathcal{C}$: its objects are the morphisms in $\mathcal{C}$, and a morphism from $f : x \to y$ to $g : z \to t$ is a commutative diagram:

this is to be compared with the usual *arrow category*, for which the morphisms are commutative squares with u and v going in the "same direction":

hence the name *twisted* arrow category. In the ∞-categorical context, we are going to take the reverse approach: we are first going to define the twisted arrow ∞-category $\mathrm{Tw}\,(\mathcal{C})$ associated to an ∞-category $\mathcal{C}$, together with a cocartesian fibration $\mathrm{Tw}\,(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, and by fact 6.4.27 this will induce a functor

$$F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

such that $F(x,y) \simeq \mathrm{Hom}_{\mathcal{C}}(x,y)$.

Definition 6.5.1 ([Lur18, Tag 0161]). Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. The *join* of $\mathcal{C}$ and $\mathcal{D}$, written $\mathcal{C} \star \mathcal{D}$, is the category whose objects are the disjoint union of those of $\mathcal{C}$ and $\mathcal{D}$, and such that

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & \text{if } x,y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(x,y) & \text{if } x,y \in \mathcal{D} \\ \{*\} & \text{if } x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing & \text{if } x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$

The join defines a functor $-\star- : \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$.

Definition 6.5.2. Let $\mathcal{C}$ be a simplicial set. The *twisted arrow* simplicial set of X, denoted $\mathrm{Tw}\,(X)$, is the simplicial set defined by

$$\mathrm{Tw}\,(X)_n := \mathrm{Hom}_{\mathrm{sSet}}\,(\mathrm{N}\,([n]^{\mathrm{op}} \star [n]), X).$$

The canonical maps

$$\Delta^n = \mathrm{N}\,([n]) \hookrightarrow \mathrm{N}\,([n]^{\mathrm{op}} \star [n])$$
$$(\Delta^n)^{\mathrm{op}} = \mathrm{N}\,([n]^{\mathrm{op}}) \hookrightarrow \mathrm{N}\,([n]^{\mathrm{op}} \star [n])$$

induces by precomposition two maps

$$\mathrm{Tw}\,(X)_n \simeq \mathrm{Hom}_{\mathrm{sSet}}\,(\mathrm{N}\,([n]^{\mathrm{op}} \star [n]), X) \to \mathrm{Hom}_{\mathrm{sSet}}\,(\mathrm{N}\,([n]), X) \simeq X_n$$
$$\mathrm{Tw}\,(X)_n \simeq \mathrm{Hom}_{\mathrm{sSet}}\,(\mathrm{N}\,([n]^{\mathrm{op}} \star [n]), X) \to \mathrm{Hom}_{\mathrm{sSet}}\,(\mathrm{N}\,([n]^{\mathrm{op}}), X) \simeq (X^{\mathrm{op}})_n$$

functorially in $[n] \in \Delta$, thus yielding by projection a *domain* and *codomain* maps

$$\mathrm{cod} : \mathrm{Tw}\,(X) \to X$$
$$\mathrm{dom} : \mathrm{Tw}\,(X) \to X^{\mathrm{op}}$$

Intuitively, the n-simplices in $\mathrm{Tw}\,(X)$ can be represented as commutative diagrams of the following form in X:

$$\begin{array}{ccccccc} x_0 & \longleftarrow & x_1 & \longleftarrow & \cdots & \longleftarrow & x_n \\ \scriptstyle f_0 \downarrow & & \scriptstyle f_1 \downarrow & & \downarrow & & \scriptstyle f_n \downarrow \\ y_0 & \longrightarrow & y_1 & \longrightarrow & \cdots & \longrightarrow & y_n \end{array}$$

So in particular when $n = 0$ and $n = 1$ we recover the expected objects and morphisms for twisted arrow categories.

Fact 6.5.3. *When $\mathcal{C}$ is an ∞-category, so is $\mathrm{Tw}\,(\mathcal{C})$.*

Fact 6.5.4 ([Lur18, Tag 03K5]). *Let $\mathcal{C}$ be an ∞-category, and x,y be objects of $\mathcal{C}$. Writing $\pi : \mathrm{Tw}\,(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ for the map defined by $\pi := \langle \mathrm{cod}, \mathrm{dom} \rangle$, π is a left fibration. Moreover, for every objects $x,y \in \mathcal{C}_0$ there is a homotopy equivalence*

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \simeq \pi^{-1}(x,y).$$

Definition 6.5.5 ([Lur18, Tag 046B]). The functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ induced by the straightening/unstraightening equivalence applied to the left fibration $\pi : \mathrm{Tw}\,(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of fact 6.5.4 is called the *hom functor* for $\mathcal{C}$, and written $\mathrm{Hom}_{\mathcal{C}}(-,-)$.

Moreover, hom functors can be shown to be natural with respect to the action of functors:

Fact 6.5.6 ([Lur18, Tag 046J]). *Every functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories induces a natural transformation*
$$\mathrm{Hom}_{\mathcal{C}}(-,-) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(F(-), G(-))$$
of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$.

As a corollary, the naturality squares of natural transformations can be chosen to be homotopy-coherent in the following sense:

Corollary 6.5.7. *Let $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ be a natural transformation. Then for every objects $c, c' \in \mathcal{C}_0$, the following square commutes up to natural isomorphism:*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(c, c') & \xrightarrow{\;\;G\;\;} & \mathrm{Hom}_{\mathcal{D}}(Gc, Gc') \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \alpha'_c \circ -} \\
\mathrm{Hom}_{\mathcal{D}}(Fc, Fc') & \xrightarrow[\;\;-\circ\alpha_c\;\;]{} & \mathrm{Hom}_{\mathcal{D}}(Fc, Gc')
\end{array}
$$

On vertices, this recover the fact that every $f : c \to c'$ induces a homotopy-commutative square:

$$
\begin{array}{ccc}
Fc & \xrightarrow{\;\;Ff\;\;} & Fc' \\
{\scriptstyle \alpha_c}\downarrow & & \downarrow{\scriptstyle \alpha'_c} \\
Gc & \xrightarrow[\;\;Gf\;\;]{} & Gc'
\end{array}
$$

Proof. Write α as a functor $\alpha : \Delta^1 \times \mathcal{C} \to \mathcal{D}$. Let $c, c' \in \mathcal{C}_0$. Applying fact 6.5.6 to α, we get in particular a natural transformation
$$\theta : \mathrm{Hom}_{\Delta^1 \times \mathcal{C}}((-,c),(-,c')) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(\alpha(-,c), \alpha(-,c'))$$
of functors $(\Delta^1)^{\mathrm{op}} \times \Delta^1 \to \mathcal{S}$. The usual naturality square for θ applied to the unique morphism $f : (1,0) \to (0,1)$ in $(\Delta^1)^{\mathrm{op}} \times \Delta^1$ precisely gives the desired result. $\qquad\square$

Having defined hom functors, we can now define representable functors as in the 1-categorical case, and state the Yoneda lemma.

Definition 6.5.8. Let $\mathcal{C}$ be an essentially small ∞-category. The ∞-category $\mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is called the ∞-category of *presheaves* on $\mathcal{C}$, and we write
$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

Remark 6.5.9 ([Lur09, Ex 5.4.1.8]). If $\mathcal{C}$ is an essentially small ∞-category, then $\mathcal{P}(\mathcal{C})$ is locally small.

Definition 6.5.10. Let $\mathcal{C}$ be an essentially small ∞-category. The hom functor $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ induces by cartesian closedness of sSet a functor
$$\yen_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(\mathcal{C})$$
called the *Yoneda embedding* for $\mathcal{C}$.

Definition 6.5.11. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a presheaf, $x \in \mathcal{C}_0$ an object of $\mathcal{C}$, and $\alpha \in F(x)_0$, identified with a morphism $\alpha : \Delta^0 \to F(x)$ in sSet. We say that α exhibit F as *represented* by x (or that (x, α) represents F) if, for every object $y \in \mathcal{C}_0$, the composite map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{S}}(F(x), F(y)) \xrightarrow{-\circ\alpha} \mathrm{Hom}_{\mathcal{S}}(\Delta^0, F(y)) \xrightarrow{\simeq} F(y)$$

is a homotopy equivalence. F is said to be *representable* if there exists an object $x \in \mathcal{C}_0$ a vertex $\alpha \in F(x)_0$ such that (x, α) represents F.

Dually, when $F : \mathcal{C} \to \mathcal{S}$, we speak of F being *corepresented* by (x, α), and of F being *corepresentable*.

Given an object $x \in \mathcal{C}_0$, the functor $\text{よ}(x) = \mathrm{Hom}(-, x) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is called the *functor represented by x*. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is said to be *representable* if it naturally isomorphic to one of the form $\text{よ}(x)$ for some object $x \in \mathcal{C}_0$. Dually, a functor $F : \mathcal{C} \to \mathcal{S}$ is said to be *corepresentable* if it is representable when seen as a functor $F : (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{S}$.

Fact 6.5.12 ([Lur18, Tag 038J]). *Let $\mathcal{C}$ be a locally small ∞-category, and let $x \in \mathcal{C}_0$. Then the presheaf $\text{よ}_{\mathcal{C}}(x) = \mathrm{Hom}_{\mathcal{C}}(-, x)$ is represented by (x, id_x). Moreover, a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is representable by x if and only if it is naturally isomorphic to $\text{よ}_{\mathcal{C}}(x)$.*

Fact 6.5.13 (*Yoneda lemma* [Lur18, Tag 03M5]). *Let $\mathcal{C}$ be a locally small ∞-category and $F, G : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be presheaves on $\mathcal{C}$ such that F is represented by (x, α). The composite functor*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, G) \xrightarrow{\mathrm{ev}_x} \mathrm{Hom}_{\mathcal{S}}(F(x), G(x)) \xrightarrow{-\circ\alpha} \mathrm{Hom}_{\mathcal{S}}(\Delta^0, G(x)) \xrightarrow{\simeq} G(x)$$

In particular, there is a natural isomorphism of presheaves

$$G \simeq \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\text{よ}(x), G).$$

As a corollary, we have:

Fact 6.5.14 (*Yoneda lemma, weak form* [Lur18, Tag 03NJ]). *The Yoneda embedding functor $\text{よ}_{\mathcal{C}} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is fully faithful.*

Remark 6.5.15. In particular, by fact 6.2.25, a morphism $f : x \to y$ in $\mathcal{C}$ is an isomorphism in $\mathcal{C}$ if and only if for every $z \in \mathcal{C}_0$, the functor

$$\mathrm{Hom}_{\mathcal{C}}(z, x) \to \mathrm{Hom}_{\mathcal{C}}(z, y)$$

is a homotopy equivalence of ∞-groupoids.

Corollary 6.5.16. *Write $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \mathcal{P}(\mathcal{C})$ for the full subcategory on representable presheaves. Then the restricted Yoneda embedding $\text{よ}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is an equivalence of ∞-categories.*

6.6 Limits and colimits

We now turn our attention to limits and colimits in the ∞-categorical setting. We can readily define initial and terminal objects using hom spaces and the idea that *unicity* is incarnated in the homotopical setting by *contractibility*:

Definition 6.6.1. Let $\mathcal{C}$ be an ∞-category, and let x be an object of $\mathcal{C}$. We say x is *initial* if for every $y \in \mathcal{C}_0$, the ∞-groupoid $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is contractible. Dually, we say that x is *terminal* if for every $y \in \mathcal{C}_0$, $\mathrm{Hom}_{\mathcal{C}}(y, x)$ is contractible.

Notice the similarity with definition 5.3.1.

Example 6.6.2. Let $\mathcal{C}$ be a category. An object x in $\mathcal{C}$ is terminal (respectively, initial) in $\mathrm{N}\,(\mathcal{C})$ if and only if it is terminal (respectively, initial) in $\mathcal{C}$.

More general limits and colimits are generally defined in the 1-categorical setting either using *representable functors* or as initial/terminal objects in categories of *cones*. Both of these approaches can be adapted to the ∞-categorical setting and they yield equivalent notions. We chose the former in our exposition:

Definition 6.6.3. Let $\mathcal{C}$ be an ∞-category and K a simplicial set. A morphism of simplicial sets $u : K \to \mathcal{C}$ is called a *diagram*.

Given $x \in \mathcal{C}_0$, the *constant diagram* at x, written $\underline{x} : K \to \mathcal{C}$ is the map obtained as the composition:

$$ K \longrightarrow \Delta^0 \xrightarrow{\ x\ } \mathcal{C} $$

Definition 6.6.4. Let $u : K \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$. A *limit* of u is an object $x \in \mathcal{C}_0$ together with a natural transformation $\alpha : \underline{x} \to u$ such that for every $y \in \mathcal{C}_0$, the composition

$$ \mathrm{Hom}_{\mathcal{C}}(y, x) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}\,(K,\mathcal{C})}(\underline{y}, \underline{x}) \xrightarrow{\ \alpha \circ -\ } \mathrm{Hom}_{\mathrm{Fun}\,(K,\mathcal{C})}(\underline{y}, u) $$

is a homotopy equivalence of ∞-groupoids. We also say that α *exhibits* x as a limit of u.

Dually, we say that a natural transformation $\alpha : u \to \underline{x}$ exhibits x as a *colimit* of u if for every $y \in \mathcal{C}_0$, the composition

$$ \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}\,(K,\mathcal{C})}(\underline{x}, \underline{y}) \xrightarrow{\ - \circ \alpha\ } \mathrm{Hom}_{\mathrm{Fun}\,(K,\mathcal{C})}(u, \underline{y}) $$

is a homotopy equivalence.

Definition 6.6.5. An ∞-category is said to be *complete* (resp. *cocomplete*) if it admits limits (resp. colimits) of all diagrams $u : K \to \mathcal{C}$ for K a small simplicial set.

All results about limits hold dually for colimits, so we will generally only state one of the two.

Fact 6.6.6 ([Lur18, Tag 02W3])**.** *Let $u : K \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$, $f : x \to y$ be a morphism in $\mathcal{C}$ and $\alpha : \underline{y} \to u$ be a natural transformation. Then any two of the following three statements imply the third:*

- *α exhibits y as a limit of u,*

- *$\alpha \circ \underline{f}$ exhibits x as a limit of u,*

- *$f : x \to y$ is an isomorphism in $\mathcal{C}$.*

In particular, limits are uniquely defined up to isomorphism (and similarly for colimits).

Definition 6.6.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, $u : K \to \mathcal{C}$ a diagram, and $(x, \alpha : \underline{x} \to u)$ a limit of u in $\mathcal{C}$. We say that F *preserves the limit* x of u if the natural transformation $F(\alpha)$ exhibits $F(x)$ as a limit of $F \circ u$.

Definition 6.6.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and K be a simplicial set. We say that F *preserves K-indexed limits* if for every diagram $u : K \to \mathcal{C}$ that admits a limit, F preserves that limit.

Fact 6.6.9 ([Lur18, Tag 0393]). *If $F : \mathcal{C} \to \mathcal{D}$ preserves a limit (x, α) of $u : K \to \mathcal{C}$, then its preserves all limits of u.*

In chapter 7, we will be interested in specific instances of shapes for indexing colimits, so we detail them now:

Definition 6.6.10. A simplicial set X is said to be *discrete* if it is isomorphic to $N(X_0)$, where the set X_0 is seen as a category with only identity morphisms. Limits of diagrams indexed by discrete simplicial sets are called *products* or cartesian products, while colimits of diagrams indexed by discrete simplicial sets are called *coproducts*.

Definition 6.6.11. A simplicial set X is said to be *finite* if it has a finite number of non-degenerate simplices. An ∞-category $\mathcal{C}$ is said to be *finite* if there exists a weak categorical equivalence $X \simeq \mathcal{C}$, with X a finite simplicial set.

The limits (resp. colimits) of diagrams indexed by finite simplicial sets are called *finite limits* (resp. *finite colimits*). When the indexing simplicial sets are moreover discrete, we speak of *finite products* and *finite coproducts*.

Example 6.6.12. A discrete simplicial set X is finite if and only if X_0 is a finite set.

Fact 6.6.13. *Let X be a discrete simplicial set, and $u : X \to \infty\mathrm{Cat}$ a family of ∞-categories. Then u admits a limit computed as the cartesian product in sSet, and a colimit computed as a the coproduct in sSet. In particular, an ∞-category $\mathcal{C}$ together with functors $\mathcal{C} \to u(x)$ for every $x \in X_0$ is a limit of u if and only if the induced functor $\mathcal{C} \to \prod_{x \in X_0} u(x)$ is an equivalence of ∞-categories (and similarly for colimits).*

Other limits and colimits in $\infty\mathrm{Cat}$ are generally not computed as in sSet.

Definition 6.6.14 ([Lur18, Tag 02P8]). An ∞-category $\mathcal{C}$ is said to be *filtered* (or $\aleph_0$-filtered) if for every diagram $u : X \to \mathcal{C}$ with X a finite simplicial set, there exists $x \in \mathcal{C}_0$ and a natural transformation $\alpha : u \to \underline{x}$.

Let $\kappa > \aleph_0$ be a cardinal. $\mathcal{C}$ is said to be *κ-filtered* if for diagram $u : X \to \mathcal{C}$ with X a κ-small simplicial set, there exists $x \in \mathcal{C}_0$ and a natural transformation $\alpha : u \to \underline{x}$.

Fact 6.6.15. *Let $\mathcal{C}$ be a filtered category in the sense of definition 3.11.21, then $N(\mathcal{C})$ is a filtered ∞-category in the sense of definition 6.6.14.*

As we explained in section 3.11.2 in the 1-categorical context, filtered diagrams generalize to categories the notion of directed subsets in a poset. Another possible generalization is that of *sifted diagrams* (definition 3.11.22), whose ∞-categorical variant is a bit more technical to define, as it relies on the notion of cofinal map of simplicial sets.

Definition 6.6.16. A morphism $f : X \to Y$ of simplicial sets is said to be *right cofinal* if for every right fibration $q : \widetilde{Y} \to Y$, the precomposition map

$$\mathrm{Fun}_{/Y}(B, \widetilde{B}) \to \mathrm{Fun}_{/Y}(X, \widetilde{B})$$

is a homotopy equivalence of ∞-groupoids (using the notation of definition 6.4.30). Note that the two ∞-categories above are ∞-groupoids because $q : \widetilde{Y} \to Y$ is a right fibration.

F is said to be *left cofinal* if $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is right cofinal.

Fact 6.6.17 ([Lur18, Tag 02PJ]). *An ∞-category $\mathcal{C}$ is filtered if and only if for every finite simplicial set X, the diagonal functor $\mathcal{C} \to \mathrm{Fun}(X, \mathcal{C})$ is right cofinal. $\mathcal{C}$ is κ-filtered if and only if for every κ-small simplicial set X, the diagonal functor $K \to \mathrm{Fun}(X, K)$ is right cofinal.*

Cofinal morphisms are usually used to ease to computation of (co)limits:

Fact 6.6.18 ([Lur18, Tag 02XR])**.** *Let $f : X \to Y$ be a right cofinal morphism of simplicial sets, $\mathcal{C}$ and ∞-category, and $u : Y \to \mathcal{C}$ a diagram. Then for every $c \in \mathcal{C}_0$ and natural transformation $\alpha : u \to \underline{c}$, α exhibits c as a colimit of u if and only if $\alpha \circ f$ exhibits c as a colimit of $u \circ f$.*

   Note that this can also be used as a definition of right cofinality [Lur18, Tag 03E7].

Definition 6.6.19 ([Lur18, Tag 02QE][Lur09, Def 5.5.8.1])**.** A simplicial set K is said to be *sifted* if for every discrete finite simplicial set X, the diagonal map $K \to \mathrm{Fun}\,(X, K)$ is right cofinal, or equivalently if K is non-empty and the diagonal map $K \to K \times K$ is right cofinal.

In particular, every (κ-)filtered ∞-category is sifted.

Remark 6.6.20. If $\mathcal{C}$ is a sifted category in the sense of definition 3.11.22, then $\mathrm{N}\,(\mathcal{C})$ is in general not sifted in the above sense (as opposed to the filtered case, fact 6.6.15). For instance, the indexing category for reflexive coequalizers (the full subcategory of Δ^{op} spanned by $[0]$ and $[1]$):

$$[1] \underset{\underset{\delta_1}{\longrightarrow}}{\overset{\overset{\delta_0}{\longrightarrow}}{\longleftarrow \sigma_0 \longrightarrow}} [0]$$

(with $\delta_0 \circ \sigma_0 = \delta_1 \circ \sigma_0 = \mathrm{id}_{[0]}$) is the prototypical example of a sifted category that is not filtered, but its nerve is not a sifted ∞-category. However, the ∞-category $\mathrm{N}\,\Delta^{\mathrm{op}}$ itself is sifted [Lur18, Tag 02QP], and is actually the prototypical example of a sifted ∞-category that is not filtered.

Definition 6.6.21. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that:

F preserves	if it preserves K-indexed colimits for every
finite colimits	finite simplicial set K
finite coproducts	discrete finite simplicial set K
small coproducts	discrete small simplicial set K
(κ-)*filtered colimits*	(κ-)filtered simplicial set K
sifted colimits	sifted simplicial set K
small colimits	small simplicial set K
all colimits	simplicial set K

and dually for limits.

   A functor that preserves small limits is also said to be *continuous*, and *cocontinuous* if it preserves small colimits. We write $\mathrm{Fun}_{\mathrm{cc}}\,(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ on the cocontinuous functors.

Remark 6.6.22. Beware that in [Lur09], the term *κ-continuous* is instead used to denote functors that preserve κ-filtered colimits.

Fact 6.6.23. *The ∞-categories $\mathcal{S}$ and $\infty\,\mathrm{Cat}$ are complete and cocomplete ([Lur18, Tag 05QQ],[Lur18, Tag 02VF],[Lur18, Tag 02TL],[Lur18, Tag 02V1]).*

   Moreover, there are explicit formulas to compute limits and colimits in $\mathcal{S}$ and $\infty\,\mathrm{Cat}$ ([Lur09, Cor 3.3.3.2, Cor 3.3.4.3]). Here we only remark how to compute limits and colimits indexed by ∞-groupoids, and how this links back to homotopy type theory.

Fact 6.6.24. *Let $u : X \to \mathcal{S}$ be a diagram of ∞-groupoids indexed by an ∞-groupoid X, and let $p : \widetilde{X} \to X$ be the Kan fibration associated to u through the straightening/unstraightening equivalence (fact 6.4.27). Then $\widetilde{X}$ is a colimit of u, and $\mathrm{Fun}_{/X}(X, \widetilde{X})$ is a limit of u.*

Proof idea. This follows from the more general results of sections 3.3.3-4 in [Lur09], but there is a more direct argument using fact 6.4.27: letting Y be any ∞-groupoid, we have a series of homotopy equivalences:

$$\mathrm{Hom}_{\mathrm{Fun}(X,\mathcal{S})}(u, Y) \simeq \mathrm{Hom}_{\mathcal{S}/X}((p : \widetilde{X} \to X), (\pi_1 : X \times Y \to X))$$
$$\simeq \mathrm{Hom}_{\mathcal{S}}(\widetilde{X}, Y)$$

exhibiting $\widetilde{X}$ as a colimit of u in $\mathcal{S}$. Similarly, there is a series of homotopy equivalences

$$\mathrm{Hom}_{\mathrm{Fun}(X,\mathcal{S})}(Y, u) \simeq \mathrm{Hom}_{\mathcal{S}/X}((\pi_1 : X \times Y \to X), (p : \widetilde{X} \to X))$$
$$\simeq \mathrm{Hom}_{\mathcal{S}}(Y, \mathrm{Fun}_{/X}(X, \widetilde{X}))$$

exhibiting $\mathrm{Fun}_{/X}(X, \widetilde{X})$ as a limit of u in $\mathcal{S}$. □

Remark 6.6.25. Combining fact 6.6.24 with remark 6.4.29, we see that the dependent sum and the dependent products of a family of types $B : A \to \mathcal{U}$ in homotopy type theory can respectively be thought of as the colimit and the limit of F, where F is thought of as a diagram $F : A \to \mathcal{S}$. This observation will be key to reconciling the definition of homotopy multisets in homotopy type theory (definition 5.7.6) with the *free commutative monoids* in the cartesian symmetric monoidal ∞-category $\mathcal{S}$ (remark 6.11.23).

An important class of examples of colimits are quotients by group actions, and in some situations they can be computed quite explicitly in $\mathcal{S}$ and $\infty\,\mathrm{Cat}$.

Definition 6.6.26. Given a group G, we write $\mathrm{B}\,G$ for the groupoid with one object $*$ and whose (iso)morphisms are the elements of G, and composition is given by the group law in G. We write $\mathbf{B}G := \mathrm{N}(\mathrm{B}\,G)$ for the nerve of $\mathrm{B}\,G$. Given an ∞-category $\mathcal{C}$, an *action of the group G on an object* $X \in \mathcal{C}_0$ is a functor $F : \mathbf{B}G \to \mathcal{C}$ such that $F(*) = X$.

Definition 6.6.27. Let $F : \mathbf{B}G \to \mathcal{C}$ be an action of the group G on an object $X := F(*)$ in an ∞-category $\mathcal{C}$. A *homotopy quotient* of X by G, denoted $X /\!\!/ G$, is a colimit of the diagram F.

Note that homotopy quotients are unique up to a contractible space of choices whenever they exist, since they are defined as being colimits. The following follows from standard results from the theory of simplicial model categories, together with the fact that *homotopy colimits* in model categories correspond to ∞-categorical colimits in their localizations ([Lur09, Thm 4.2.4.1]).

Fact 6.6.28. *Let $F : \mathrm{B}\,G \to \mathrm{Kan}$ be a functor (in the ordinary, 1-categorical sense), and suppose that the corresponding action of G on $X := F(*)$ is* free, *in the sense that the induced actions of G on the sets X_n is a free group action for every $n \in \mathbb{N}$. Then the ordinary quotient X/G (i.e. the ordinary colimit of F) is a Kan complex, and it is moreover a homotopy quotient of X by G, i.e. a colimit of the induced functor $\widetilde{F} : \mathbf{B}G \to S$ (this follows from the fact that such a diagram is* projectively cofibrant, *see [GJ09, Cor 2.10]). The same is true for a functor $F : \mathrm{B}\,G \to \mathrm{qCat}$ and the corresponding $\widetilde{F} : \mathbf{B}G \to \infty\,\mathrm{Cat}$.*

Fact 6.6.29. *Given a group G, write $\mathrm{E}\,G$ for the discrete groupoid on the set $|G|$ (i.e. the groupoid having $|G|$ as objects and a unique isomorphism between any pair of objects), and $\mathbf{E}G := \mathrm{N}(\mathrm{E}\,G)$ for its nerve. G acts freely by (left) multiplication on $\mathrm{E}\,G$, so given any action of G on a Kan complex X, we get a free*

action of G on **E***G* × *X. Moreover, since* E *G is a contractible groupoid,* **E***G* × *X and X are equivalent, so their homotopy quotients by G are equivalent. In particular, this means that* (**E***G* × *X*)/*G is a homotopy quotient of X by G (see also* [*Rie14, Ex 6.4.11*])*, so we can define:*

$$X \mathbin{/\mkern-4mu/} G := (\mathbf{E}G \times X)/G.$$

Just like in the 1-categorical setting, limits and colimits in ∞-categories of diagrams can be computed pointwise:

Fact 6.6.30 ([Lur18, Tag 02X9])**.** *Let* $\mathcal{C}$ *be an ∞-category, and K, X be simplicial sets, and* $u : K \to$ Fun $(X, \mathcal{C})$ *be a diagram. Suppose that for every vertex* $x \in X_0$*, the composite diagram*

$$K \xrightarrow{\ u\ } \text{Fun}\,(X, \mathcal{C}) \xrightarrow{\ \text{ev}_x\ } \mathcal{C}$$

admits a colimit in $\mathcal{C}$*. Then :*

1. *the diagram u admits a colimit in* Fun $(X, \mathcal{C})$,

2. *let* $f : X \to \mathcal{C}$ *and* $\alpha : \underline{u} \Rightarrow f$*. Then α exhibits f as a colimit of u if and only if for every vertex* $x \in X_0$, $\text{ev}_x \circ \alpha$ *exhibits* $\text{ev}_x \circ f$ *as a colimit of* $\text{ev}_x \circ u$ *in* $\mathcal{C}$*.*

As a corollary of , we have:

Fact 6.6.31. *Let* $\mathcal{C}$ *be an ∞-category. Then* $\mathcal{P}(\mathcal{C})$ *is complete and cocomplete.*

And just like in category theory, we can prove that $\mathcal{P}(\mathcal{C})$ is actually the *free cocompletion* of $\mathcal{C}$:

Fact 6.6.32 ([Lur18, Tag 04BE])**.** *Let* $\mathcal{C}$ *be a small ∞-category. Then for every cocomplete ∞-category* $\mathcal{D}$*, precomposition with the Yoneda embedding* よ$_{\mathcal{C}}$ *induces an equivalence of ∞-categories:*

$$\text{Fun}_{\text{cc}}\,(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow[\simeq]{-\circ\, ぇ_{\mathcal{C}}} \text{Fun}\,(\mathcal{C}, \mathcal{D})$$

6.7 Adjunctions

There are many equivalent ways to define adjunctions between ordinary categories. In the ∞-categorical setting, many different approaches have been explored by generalizing the various 1-categorical definitions, and all those approaches give equivalent notions, even though this equivalence is way more involved than in the classical story.

Fact 6.7.1 ([Cis19, Thm 6.1.23])**.** *Let* $L : \mathcal{C} \to \mathcal{D}$ *and* $R : \mathcal{D} \to \mathcal{C}$ *be functors between ∞-categories. The following are equivalent:*

1. *there exists a natural isomorphism*

$$\text{Hom}_{\mathcal{D}}(L(-), -) \simeq \text{Hom}_{\mathcal{C}}(-, R-)$$

   of functors $\mathcal{D}^{\text{op}} \times \mathcal{C} \to \mathcal{S}$*.*

2. *there exists a natural transformation* $\eta : \text{id}_{\mathcal{C}} \Rightarrow R \circ L$ *such that for every* $c \in \mathcal{C}, y \in \mathcal{D}$*, the induced functor*

$$\text{Hom}_{\mathcal{D}}(L(c), d) \xrightarrow{\ R\ } \text{Hom}_{\mathcal{C}}(RL(c), R(d)) \xrightarrow{\ -\circ\eta_c\ } \text{Hom}_{\mathcal{C}}(c, R(d))$$

   is a homotopy equivalence.

3. *there exists a natural transformation $\varepsilon : L \circ R \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that for every $c \in \mathcal{C}$, $y \in \mathcal{D}$, the induced functor*

$$\mathrm{Hom}_{\mathcal{C}}(c, R(d)) \xrightarrow{\ L\ } \mathrm{Hom}_{\mathcal{D}}(L(c), RL(d)) \xrightarrow{\ \varepsilon_d \circ -\ } \mathrm{Hom}_{\mathcal{D}}(L(c), d)$$

   is a homotopy equivalence.

4. *there exists natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to R \circ L$ and $\varepsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ and commutative triangles of natural transformations*

   in $\mathrm{Fun}\,(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}\,(\mathcal{D}, \mathcal{C})$ respectively.

Definition 6.7.2. Under the equivalent conditions of fact 6.7.1, we say that L is *left adjoint* to R, or that R is *right adjoint* to L, or that there is an *adjunction $L \dashv R$*. Moreover:

1. if $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$ is a natural transformation as in item 2 of fact 6.7.1, we say that η is a *unit* for the adjunction $L \dashv R$,

2. if $\varepsilon : L \circ R \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural transformation as in item 3 of fact 6.7.1, we say that ε is a *counit* for the adjunction $L \dashv R$,

3. if η and ε are natural transformations as in item 4 of fact 6.7.1, we say that (η, ε) is a *unit-counit pair* for the adjunction $L \dashv R$.

Remark 6.7.3. It may come as a surprise that item 1 of fact 6.7.1 gives rise to a well-behaved notion of adjunction for functors between ∞-categories: one might expect for instance the need for higher coherences between the specified commutative triangles, themselves possibly subject to higher coherences. It is actually possible to give such a homotopy-coherent definition of adjunctions, but it has been shown by Riehl and Verity in [RV16] that the data of item 1 can always be extended to such higher-coherences. This is similar to the situation in homotopy type theory where the notion of equivalence of types could be defined by only referring to low-dimensional paths.

The terminology for units, counits and unit-counit pairs is justified by the following fact:

Fact 6.7.4 ([Lur18, Tag 02FX]). *We have the following:*

1. *If η is a unit for $L \dashv R$, then there exists $\varepsilon : L \circ R \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that (η, ε) is a unit-counit pair for $L \dashv R$.*

2. *If ε is a counit for $L \dashv R$, then there exists $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$ such that (η, ε) is a unit-counit pair for $L \dashv R$.*

3. *If (η, ε) is a unit-counit pair for $L \dashv R$, then η is a unit for $L \dashv R$ and ε is a counit for $L \dashv R$.*

Remark 6.7.5 ([Lur18, Tag 02ER]). It readily follows from definition 6.7.2 that $L \dashv R$ if and only if $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$. Moreover, if $\eta : \mathrm{id}_{\mathcal{C}} \to R \circ L$ is a unit for $L \dashv R$, then $\eta^{\mathrm{op}} : R^{\mathrm{op}} \circ L^{\mathrm{op}} \to \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$ is a counit for $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$, and symmetrically for counits.

We now review some standard facts about adjunctions between ∞-categories. First, equivalences of ∞-categories are adjunctions:

Fact 6.7.6 ([Lur18, Tag 02EV][RV22, Prop 2.1.12]). *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and $G : \mathcal{D} \to \mathcal{C}$ and inverse of F, i.e. such there are natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \simeq G \circ F$ and $\varepsilon : F \circ G \simeq \mathrm{id}_{\mathcal{D}}$. Then $F \dashv G$, $G \dashv F$, η is a unit for $F \dashv G$ and ε is a counit for $F \dashv G$. In general however (η, ε) is not a unit-counit pair for $F \dashv G$.*

Adjoint functors are unique up to natural isomorphism:

Fact 6.7.7 ([Lur18, Tag 02F3], [RV22, Prop 2.1.10]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, with a right adjoint $R : \mathcal{D} \to \mathcal{C}$. A functor $R' : \mathcal{D} \to \mathcal{C}$ is right adjoint to F if and only if $R \simeq R'$.*

Adjoint functors can be composed:

Fact 6.7.8 ([Lur18, Tag 02ES]). *Given a diagram of ∞-categories and adjoint functors*

$$\mathcal{C} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{D} \underset{R'}{\overset{L'}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}$$

we have $L' \circ L \dashv R \circ R'$.

Full faithfulness of an adjoint functor can be checked at the level of the (co)unit:

Proposition 6.7.9. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be adjoint functors with a unit-counit pair (η, ε).*

- *L is fully faithful if and only if η is a natural isomorphism.*

- *R is fully faithful if and only if ε is a natural isomorphism.*

Proof. Applying corollary 6.5.7 to the natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$, we get the following homotopy-commutative rectangle:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(c, c') \xrightarrow{\ L\ } \mathrm{Hom}_{\mathcal{D}}(L(c), L(c')) \xrightarrow{\ R\ } & \mathrm{Hom}_{\mathcal{C}}(RL(c), RL(c')) \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}}} \qquad\qquad {\scriptstyle \simeq} \qquad\qquad & \Big\downarrow{\scriptstyle -\circ \eta_c} \\
\mathrm{Hom}_{\mathcal{C}}(c, c') \xrightarrow[\ \eta'_c \circ - \ ]{} & \mathrm{Hom}_{\mathcal{C}}(c, RL(c'))
\end{array}
$$

So that the action of L on hom spaces

$$L : \mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(L(c), L(c'))$$

is naturally isomorphic to the composition

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \xrightarrow{\ \eta'_c \circ - \ } \mathrm{Hom}_{\mathcal{C}}(c, RL(c')) \xrightarrow{\ \simeq \ } \mathrm{Hom}_{\mathcal{D}}(L(c), L(c'))$$

In particular, the former is a homotopy equivalence for every c, c' if and only if $(\eta'_c \circ -) : \mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{C}}(c, RL(c'))$ is a homotopy equivalence. By remark 6.5.15, this holds for every c if and only if $\eta'_c : c' \to RL(c')$ is an isomorphism in $\mathcal{C}$. Finally, η'_c is an isomorphism for every c' if and only if η is a natural isomorphism (fact 6.2.25), which concludes the proof that L is fully faithful if and only if η is a natural isomorphism.

The case of R and ε follows by taking opposites. $\square$

Remark 6.7.10. Since being fully faithful is an intrinsic property of the functor that does not depend on the choice of a unit, then *any* unit for the adjunction $L \dashv R$ is a natural isomorphism if and only if *every* unit is, and similarly for counits.

Proposition 6.7.11. *Let* $L : \mathcal{C} \to \mathcal{D}$ *be left adjoint to* $R : \mathcal{D} \to \mathcal{C}$ *with a unit-counit pair* (η, ε). *The following are equivalent :*

1. *L is an equivalence of ∞-categories,*

2. *R is an equivalence of ∞-categories,*

3. *η and ε are natural isomorphisms,*

Proof. Assertion (3) implies that L and R are inverses of each other up to natural isomorphism, so they form an equivalence of ∞-categories. Let's show (1) $\implies$ (2). If L is an equivalence, its has an inverse up to natural isomorphism R', which by fact 6.7.6 is a right adjoint to L. By fact 6.7.7, this implies that $R \simeq R'$, so that R is also an inverse to L, which implies that R is itself an equivalence. A dual argument show that (2) $\implies$ (1). Finally, if both L and R are equivalences, in particular they are fully faithful, so by proposition 6.7.9 η and ε are natural isomorphisms. $\square$

Let $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$ be adjoint functors, and $\varepsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ be a counit for $L \dashv R$. Given an object $d \in \mathcal{D}_0$, the composite natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(-, R(d)) \xrightarrow{\quad L \quad} \mathrm{Hom}_{\mathcal{D}}(L(-), LR(d)) \xrightarrow{\varepsilon_d \circ -} \mathrm{Hom}_{\mathcal{D}}(L(-), d)$$

is a natural isomorphism of presheaves $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, so the presheaf $\mathrm{Hom}_{\mathcal{D}}(L(-), d)$ on $\mathcal{C}$ is represented by $(R(d), \varepsilon_d)$. It turns out that this representability property is sufficient to guarantee that L has a right adjoint:

Fact 6.7.12 ([Lur18, Tag 02FV]). *Let* $L : \mathcal{C} \to \mathcal{D}$ *be a functor between ∞-categories. Suppose that for every object* $d \in \mathcal{D}_0$, *the presheaf*

$$\mathrm{Hom}_{\mathcal{D}}(L(-), d) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$$

is represented by some (c_d, α_d), *with* $c_d \in \mathcal{C}_0$ *and* $\alpha_d : L(c_d) \to d$. *Then L admits a right adjoint* $R : \mathcal{D} \to \mathcal{C}$ *with a counit* $\varepsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ *such that for every* $d \in \mathcal{D}_0$, *there is an isomorphism* $f_d : c_d \simeq R(d)$ *and a commutative triangle*

$$
\begin{array}{ccc}
L(c_d) & \xrightarrow[\simeq]{L(f_d)} & LR(d) \\
& {\scriptstyle \alpha_d} \searrow \quad \swarrow {\scriptstyle \varepsilon_d} & \\
& d &
\end{array}
$$

in $\mathcal{D}$.

Dually, if $R : \mathcal{D} \to \mathcal{C}$ *is a functor such that for every* $c \in \mathcal{C}_0$, *the copresheaf*

$$\mathrm{Hom}_{\mathcal{C}}(c, R(-)) : \mathcal{D} \to \mathcal{S}$$

is corepresentable, then R admits a left adjoint L with a unit $\eta : \mathrm{id}_{\mathcal{C}} \to R \circ L$ *that satisfies a dual condition to that above.*

Fact 6.7.12 is very useful to construct adjoints in practice. For instance, it can be used to show the functoriality of limits and colimits:

Fact 6.7.13 ([Lur18, Tag 05Q9]). *Let $\mathcal{C}$ be an ∞-category and K a simplicial sets, and write $\Delta : \mathcal{C} \to$ Fun $(K, \mathcal{C})$ for the diagonal map $x \mapsto \underline{x}$. We have:*

- *$\mathcal{C}$ admits K-indexed limits if and only if Δ has a right adjoint $\lim_K$ called the* limit functor, *in which case for every diagram $d : K \to \mathcal{C}$, $\lim d$ is a colimit of d in $\mathcal{C}$.*

- *$\mathcal{C}$ admits K-indexed colimits if and only if Δ has a left adjoint colim_K called the* colimit functor, *in which case for every diagram $d : K \to \mathcal{C}$, $\operatorname{colim} d$ is a colimit of d in $\mathcal{C}$.*

Proof. This is fact 6.7.12 instantiated in the case $L := \Delta$ or $R := \Delta$ respectively. □

Note that by fact 6.7.7, the functors $\lim$ and colim are unique up to natural isomorphism when they exist.

Remark 6.7.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and suppose that both $\mathcal{C}$ and $\mathcal{D}$ admit K-indexed limits. If F preserves K-indexed limits, then the following square of functors commutes up to natural isomorphism:

$$\begin{array}{ccc}
\text{Fun}\,(K, \mathcal{C}) & \xrightarrow{F \circ -} & \text{Fun}\,(K, \mathcal{D}) \\
{\scriptstyle \lim} \downarrow & & \downarrow {\scriptstyle \lim} \\
\mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D}
\end{array}$$

The converse if *false* in general: *commuting* with limits does not imply *preserving limits*. Caccamo and Winskel have investigated sufficient conditions for this converse to hold in [CW05a] in the 1-categorical context. In particular they have shown that if $\mathcal{C}$ and $\mathcal{D}$ are categories with terminal objects, and $F : \mathcal{C} \to \mathcal{D}$ preserves terminal objects, then it preserves K-indexed limits if and only if it commutes with K-indexed limits in the above sense [CW05a, Thm 4.6]. However, their proof relies on the fact that when K is a *connected* category, $\operatorname{colim}_K \underline{x} \simeq x$, a result that does not hold for ∞-categorical limits, where K generally needs to be *weakly contractible* for this to hold. To my knowledge, generalizing Caccamo and Winskel's result to ∞-categorical (co)limits is still an open problem.

Fact 6.7.12 can also be used to prove the following enhancement to fact 6.7.7:

Fact 6.7.15 ([Lur09, Prop 5.2.6.2]). *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, write $\text{Fun}^{\text{L}}\,(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\text{Fun}\,(\mathcal{C}, \mathcal{D})$ spanned by the functors which are left adjoints, and $\text{Fun}^{\text{R}}\,(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\text{Fun}\,(\mathcal{C}, \mathcal{D})$ spanned by the functors which are right adjoints. There is a canonical equivalence of ∞-categories:*

$$\text{Fun}^{\text{L}}\,(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{R}}\,(\mathcal{D}, \mathcal{C})^{\text{op}}$$

Proof. We follow the argument given in the proof of [Lur09, Prop 5.2.6.2]. Post-composition with the yoneda embedding $\text{よ}_{\mathcal{D}}$ induces a fully faithful functor

$$\text{Fun}\,(\mathcal{C}, \mathcal{D}) \xrightarrow{\text{よ}_{\mathcal{D}} \circ -} \text{Fun}\,(\mathcal{C}, \mathcal{P}(\mathcal{D})) \xrightarrow{\simeq} \text{Fun}\,(\mathcal{C} \times \mathcal{D}^{\text{op}}, \mathcal{S})$$

whose essential image consists of the functors $G : \mathcal{C} \times \mathcal{D}^{\text{op}} \to \mathcal{S}$ such that for every object $c \in \mathcal{C}_0$, the induced functor $G(c, -) : \mathcal{D}^{\text{op}} \to \mathcal{S}$ is representable.

By fact 6.7.12, the a functor $G : \mathcal{C} \times \mathcal{D}^{\text{op}} \to \mathcal{S}$ is in the essential image of the composition

$$\text{Fun}^{R}(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}\,(\mathcal{C}, \mathcal{D}) \to \text{Fun}\,(\mathcal{C} \times \mathcal{D}^{\text{op}}, \mathcal{S})$$

if, in addition to the previous condition, for every object $d \in \mathcal{D}_0$, the induced functor $G(-, d) : \mathcal{C} \to \mathcal{S}$ is corepresentable.

Symmetrically, this is the same as the essential image of the composition

$$\mathrm{Fun}^{\mathrm{R}}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}\right) \overset{\subseteq}{\longrightarrow} \mathrm{Fun}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}\right) \overset{\jmath_{\mathcal{C}^{\mathrm{op}}} \circ -}{\longrightarrow} \mathrm{Fun}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{P}(\mathcal{C}^{\mathrm{op}})\right) \overset{\simeq}{\longrightarrow} \mathrm{Fun}\left(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathcal{S}\right)$$

Fully faithful functors are equivalences onto their essential image by fact 6.3.11, so we get an equivalence

$$\mathrm{Fun}^{\mathrm{R}}\left(\mathcal{C}, \mathcal{D}\right) \simeq \mathrm{Fun}^{\mathrm{R}}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}\right). \tag{6.4}$$

Finally, remark 6.7.5 allows us to restrict the opposite equivalence

$$\mathrm{Fun}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}\right) \simeq \mathrm{Fun}\left(\mathcal{D}, \mathcal{C}\right)^{\mathrm{op}}$$

of remark 6.2.21 to an equivalence

$$\mathrm{Fun}^{\mathrm{R}}\left(\mathcal{D}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}\right) \simeq \mathrm{Fun}^{\mathrm{L}}\left(\mathcal{D}, \mathcal{C}\right)^{\mathrm{op}},$$

which, when composed with the equivalence 6.4 yields the desired equivalence

$$\mathrm{Fun}^{\mathrm{R}}\left(\mathcal{C}, \mathcal{D}\right) \simeq \mathrm{Fun}^{\mathrm{L}}\left(\mathcal{D}, \mathcal{C}\right)^{\mathrm{op}}.$$

$\square$

A functor $F : \mathcal{C} \to \mathcal{D}$ can be seen as a morphism in $\infty \mathrm{Cat}$, hence as a map $F : \Delta^1 \to \infty \mathrm{Cat}$. Applying fact 6.4.27, this is equivalent to the data of a cocartesian fibration $p : \mathcal{M} \to \Delta^1$. Applying fact 6.7.12, the condition of being an adjoint can be rephrased in terms of p as follows:

Fact 6.7.16 ([Lur18, Tag 02FP]). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor classified by a cocartesian fibration $p : \mathcal{M} \to \Delta^1$. Then F has a right adjoint if and only if p is also a cartesian fibration, in which case a right adjoint to F is given by the contravariant transport f^* where f is the only non-degenerate edge in Δ^1.*

Symmetrically, F is classified by a cartesian fibration $q : \mathcal{N} \to \Delta^1$, in which case F has a left adjoint if and only if q is also a cocartesian fibration, and the left adjoint is given by covariant transport $f_!$.

Corollary 6.7.17. *Write $\infty \mathrm{Cat}^{\mathrm{L}} \subseteq \infty \mathrm{Cat}$ for the full subcategory of $\infty \mathrm{Cat}$ spanned by the functors which are left adjoints, and $\infty \mathrm{Cat}^{\mathrm{R}}$ for the right adjoints. There is an equivalence*

$$\infty \mathrm{Cat}^{\mathrm{L}} \simeq \left(\infty \mathrm{Cat}^{\mathrm{R}}\right)^{\mathrm{op}}.$$

We close this section with the classical result that adjoints preserve (co)limits, of which we will state a partial converse in section 6.8 (fact 6.8.15):

Fact 6.7.18. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let $u : K \to \mathcal{C}$ be a diagram in $\mathcal{C}$.*

- *If u has a limit in $\mathcal{C}$ and F is a right adjoint, then F preserves the limit of u.*

- *If u has a colimit in $\mathcal{C}$ and F is a left adjoint, then F preserves the colimit of u.*

6.8 Presentable ∞-categories

In this section, we overview the ∞-categorical variant of locally presentable categories. The reference book on the subject in the 1-categorical setting is by far Adamek and Rosicky's [Adá94]. Since the theory of locally presentable categories is not as well known as other standard categorical notions, especially in computer science, let us mention a few motivations first.

Let $(E, \leq)$ be a poset with arbitrary joins. Then E is a complete lattice: it also admits arbitrary meets, where the meet of a subset $U \subseteq E$ is computed as the join of all the lower bounds of U. Going from posets to categories, the situation is not as nice: if a category $\mathcal{C}$ admits all colimits, it is equivalent to a poset, this is why we generally only consider categories with *small* colimits. But then, if $\mathcal{C}$ admits small colimits, it does not necessarily admit small limits too. Intuitively, if we were to try the same trick as for posets, we would try to compute the limit of a diagram $u : K \to \mathcal{C}$ as a colimit indexed by the cones $\underline{x} \Rightarrow u$ under d, but the collection of such cones is not itself small! Intuitively, a category being locally presentable means that it is *generated* by a small set of *generating objects* in a precise sense. This makes it possible to consider only cones $\underline{x} \Rightarrow u$ for x a *generating object* in the above argument, and since there is a small set of such objects, the proof actually goes through.

Now consider $f : (E, \leq) \to (F, \leq)$ a monotone map between complete lattices, and suppose that f preserves all joins (i.e. it is a suplattice morphism). Then one can define a map $g : F \to E$ by

$$ g(y) := \bigvee_{f(x) \leq y} x $$

and prove that for every $x \in E, y \in F$,

$$ f(x) \leq y \Leftrightarrow x \leq g(y), $$

which is to say that f is left adjoint to g in the posetal sense. Attempting a similar construction to show that every colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ admits a right adjoint, we get into the same issue as before, trying to construct the colimit of a non-small diagram. But if $\mathcal{C}$ is locally presentable, we can rely on generating objects to build such a colimit, and then the proof goes through: this is the content of the so-called *adjoint functor theorems*.

The term "locally presentable category" is due to Gabriel and Ulmer [GU71]. The intuition behind the terminology is the following: in a locally presentable category $\mathcal{C}$, every object is a kind of "directed union" (filtered colimit) of "presentable objects", so every object is in that sense "locally presentable". Nowadays, "presentable objects" are more commonly referred to as κ-compact objects, and in the ∞-categorical context, it is more common (following Lurie [Lur09]) to simply drop the word "locally" and speak of presentable ∞-categories, with the idea that a presentable ∞-category $\mathcal{C}$ is itself "presented" or *generated* by its κ-compact objects.

Definition 6.8.1. Let κ be a *regular cardinal*, i.e. a cardinal such that for every family of sets $(X_i)_{i \in I}$ with $\#I < \kappa$ and $\#X_i < \kappa$ for all i, then $\# \cup_i X_i < \kappa$.

Let $\mathcal{C}$ be an ∞-category that admits κ-filtered colimits. We say an object $x \in \mathcal{C}_0$ is κ-*compact* if the hom functor

$$ \mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S} $$

preserves κ-filtered colimits.

In the case $\kappa := \aleph_0$, we simply say that x is compact.

Example 6.8.2. Let $\mathcal{C}$ be an ordinary category with filtered colimits. An object x of $\mathcal{C}$ is compact if for every filtered diagram $u : K \to \mathcal{C}$, the canonical map of sets

$$ \theta : \mathrm{Hom}_{\mathcal{C}}(x, \mathrm{colim}_{k \in K} u(k)) \to \mathrm{colim}_{k \in K} \mathrm{Hom}_{\mathcal{C}}(x, u(k)) $$

is a bijection. This map is always injective, so we are really asking for θ to be surjective. In other words, x is compact if every morphism $f : x \to \operatorname{colim}_{k \in K} u(k)$ factors through a map $\overline{f} : x \to u(k)$ for some $k \in K$.

In the case $\mathcal{C} = \operatorname{Set}$, this can in turn be understood as: a set X is compact if every map from X to a directed union of sets $\cup_i Y_i$ factors through one of the Y_i, and this happens precisely when X is finite. This proves that the compact objects in Set are the finite sets.

Example 6.8.3 ([Lur09, Ex 5.3.4.2])**.** A group G is compact if and only if it is *finitely presented* as group, i.e. G can be represented as the quotient of the free group over a finite set of generators by a finite set of relations. This holds more generally for abelian groups, rings, k-algebras, R-modules, and other categories of "algebraic objects", and is the motivation for the older terminology of *κ-presented object* that can be found in [Adá94].

Example 6.8.4. If $\kappa > \aleph_0$, the κ-compact objects in $\mathcal{S}$ (resp. $\infty \operatorname{Cat}$) are the essentially κ-small ∞-groupoids (resp. ∞-categories). The compact objects in $\mathcal{S}$ are the retracts of finite ∞-groupoids (which in general are not necessarily finite) [Lur17, Warning 1.4.2.7].

We can now state the definition of presentable ∞-categories.

Definition 6.8.5. Let $\mathcal{C}$ be an ∞-category, X a collection of objects in $\mathcal{C}$, and $\mathcal{K}$ a collection of simplicial sets. We say that *X generates $\mathcal{C}$ under $\mathcal{K}$-colimits* if for every full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ containing X, and such that $\mathcal{C}'$ admits K-indexed colimits for every $K \in \mathcal{K}$, then $\mathcal{C}' = \mathcal{C}$.

Definition 6.8.6. Let $\mathcal{C}$ be a locally small ∞-category. We say that $\mathcal{C}$ is *accessible* if there exists a regular cardinal κ such that:

1. the full subcategory $\mathcal{C}^\kappa \subseteq \mathcal{C}$ spanned by the κ-compact objects is essentially small,

2. $\mathcal{C}^\kappa$ generates $\mathcal{C}$ under κ-filtered colimits.

Definition 6.8.7. An ∞-category is said to be *presentable* if it is accessible and admits small colimits.

Example 6.8.8. $\mathcal{S}$ is presentable. More generally, if $\mathcal{C}$ is a small ∞-category, $\mathcal{P}(\mathcal{C})$ is presentable [Lur09, Thm 5.5.1.1].

Example 6.8.9. $\infty \operatorname{Cat}$ is presentable (this is an instance of the vastly more general lemma 4.8.4.2 in [Lur17]).

Example 6.8.10. The nerve of a locally presentable category is a presentable ∞-category. In particular, the nerves of the categories of groups, abelian groups, rings, k-algebras,…are presentable ∞-categories.

Example 6.8.11. Let $n \geq 0$. An ∞-groupoid X is said to be an *n-groupoid* if its homotopy groups $\pi_k(X)$ are trivial for all $k > n$. The full subcategory $\mathcal{S}_{\leq n} \subseteq \mathcal{S}$ spanned by the n-groupoids is presentable.

Example 6.8.12.

A natural condition to impose on functors between presentable ∞-categories is the following:

Definition 6.8.13 ([Lur09, Def 5.4.2.5])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, with $\mathcal{C}$ an accessible ∞-category. We say that F is an *accessible functor* if it preserves κ-filtered colimits for some regular cardinal κ.

We can now state the results promised at the beginning of this section:

Fact 6.8.14 ([Lur09, Cor 5.5.2.4])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then $\mathcal{C}$ admits small limits.*

Fact 6.8.15 (The *adjoint functor theorem* [Lur09, Cor 5.5.2.9, Rmk 5.5.2.10])**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, with $\mathcal{C}$ presentable and $\mathcal{D}$ locally small.*

  1. *F has a right adjoint if and only if it preserves small colimits.*

  2. *F has a left adjoint if and only if it is accessible and preserves small limits.*

In particular, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{cc}}(\mathcal{C}, \mathcal{D})$.

The asymmetry of fact 6.8.15 is due to the asymmetry of the definition of presentable ∞-category: if $\mathcal{C}$ is presentable, $\mathcal{C}^{\mathrm{op}}$ is in general not presentable (unless $\mathcal{C}$ is a poset, see [Adá94, Thm 1.64] for the 1-categorical case).

Fact 6.8.16 ([Lur09, Prop 5.5.3.6, Prop 5.5.3.8])**.** *Let $\mathcal{C}, \mathcal{D}$ be presentable ∞-categories, and K a simplicial set. Then* $\mathrm{Fun}_{\mathrm{cc}}(\mathcal{C}, \mathcal{D})$ *and* $\mathrm{Fun}(K, \mathcal{D})$ *are presentable.*

As a corollary of fact 6.8.15 and corollary 6.7.17, we get:

Fact 6.8.17 ([Lur09, Cor 5.5.3.4])**.** *Write* Pr^{L} *for the subcategory of* $\infty\,\mathrm{Cat}_{\mathcal{U}_1}$ *spanned by the presentable ∞-categories and the cocontinuous functors, and* Pr^{R} *the subcategory of* $\infty\,\mathrm{Cat}_{\mathcal{U}_1}$ *spanned by the presentable ∞-categories and the functors which are both accessible and continuous (where* $\infty\,\mathrm{Cat}_{\mathcal{U}_1}$ *denotes the very large ∞-category for large ∞-categories). Then* $\mathrm{Pr}^{\mathrm{L}} \subseteq \infty\,\mathrm{Cat}^{\mathrm{L}}$ *and* $\mathrm{Pr}^{\mathrm{R}} \subseteq \infty\,\mathrm{Cat}^{\mathrm{R}}$ *are full subcategories, and there is an equivalence*

$$\mathrm{Pr}^{\mathrm{L}} \simeq (\mathrm{Pr}^{\mathrm{R}})^{\mathrm{op}}.$$

This can be used to compute colimits in Pr^{L}:

Fact 6.8.18 ([Lur09, Prop 5.5.3.13, Thm 5.5.3.18])**.** *The ∞-categories* Pr^{L} *and* Pr^{R} *admit small limits, and the inclusion functors* $\mathrm{Pr}^{\mathrm{L}} \subseteq \infty\,\mathrm{Cat}_{\mathcal{U}_1}$, $\mathrm{Pr}^{\mathrm{R}} \subseteq \infty\,\mathrm{Cat}_{\mathcal{U}_1}$ *preserve small limits.*

Corollary 6.8.19. Pr^{L} *(resp.* Pr^{R} *) admits small colimits, and they are computed as limits in* $\mathrm{Pr}^{\mathrm{L\,op}} \simeq \mathrm{Pr}^{\mathrm{R}}$ *(resp.* $\mathrm{Pr}^{\mathrm{R\,op}} \simeq \mathrm{Pr}^{\mathrm{L}}$ *).*

6.9 Introduction to higher algebra

In the remainder of this chapter, we review the theory of symmetric monoidal ∞-categories and commutative monoids within, monoidal ∞-categories and their monoids, as well as monads on ∞-categories as a particular instance of the latter, and the relationship between monads and adjunctions. We regroup all these concepts together as they can all be described through Lurie's unified framework of ∞-operads, which he developed in a series of articles later compiled in his book *Higher Algebra* [Lur17].

Classically, an operad O is the data for each $n \in \mathbb{N}$ of a set O_n of *operations of arity n*, together with a specification of how these operations can be composed. Any operad O then defines a notion of *O-algebra*, where an *O*-algebra in Set is the data of a set X together with a map $f_o : X^n \to X$ for every operation $o \in O_n$, and such that the composition of the maps f_o respects the specified

compositions of the operations. Operads typically come in two flavours: planar operads, which we just described, and *symmetric operads*, where the set O_n is equipped with an action of the symmetric group $\mathfrak{S}_n$ to represent the possibility of swapping the inputs. Operads can also be *colored*, i.e. the arities of operations can also specify input and output *colors*. Colored operads are also called *multicategories*, where the colors are instead named *objects* and the operations are called *multimorphisms*, with the set of multimorphisms written

$$\mathrm{Hom}_O(c_1, \ldots, c_n; d).$$

Multicategories can be used to define (symmetric) monoidal categories: indeed, consider a multicategory such that for every objects $c_1, \ldots, c_n$, there exists an object, suggestively written $c_1 \otimes \cdots \otimes c_n$, such that for every object d there is a bijection

$$\mathrm{Hom}_o(c_1, \ldots, c_n; d) \simeq \mathrm{Hom}_o(c_1 \otimes \cdots \otimes c_n; d).$$

Then the operation $\otimes$ defines a monoidal structure on the underlying category of the multicategory O: we also say that the multicategory O is *representable*.

Slogan 6.9.1. A symmetric monoidal category is a *representable multicategory*. *Being* a symmetric monoidal category is a *property* of a multicategory.

This approach to monoidal categories has been developed for instance in section 3.3 of Tom Leinster's book [Lei04].

Lurie's theory of ∞-operads and (symmetric) monoidal structures on ∞-categories is similar: in his setting, every ∞-operad $\mathcal{O}$ comes with a notion of $\mathcal{O}$-monoidal ∞-category, which are themselves particular instances of ∞-operads. We will mainly be interested in three ∞-operads:

1. the commutative ∞-operad Fin_* to define symmetric monoidal ∞-categories and their commutative monoids,

2. the associative ∞-operad Assoc to define monoidal ∞-categories and their monoids, and in particular monads,

3. the ∞-operad of left modules LM to define algebras over a monad.

The latter two will mostly be used as a blackbox in chapter 7, where we will mainly work with the adjunctions induced by a monad rather than the monad itself.

6.10 Symmetric monoidal ∞-categories

Rather than directly define ∞-operads and then symmetric monoidal ∞-categories as a special case, we will first motivate the definition of symmetric monoidal ∞-categories and then generalize it to ∞-operads. The leading idea is the following: a symmetric monoidal category is just a commutative monoid in the bicategory of categories with respect to the cartesian product, so a symmetric monoidal ∞-category should be nothing more than a commutative monoid in $\infty\,\mathrm{Cat}$ with respect to the cartesian products of ∞-categories.

In a category $\mathcal{C}$ with finite products, a commutative monoid object is usually defined as an object M of $\mathcal{C}$ together with morphisms $m : M \times M \to M$ for "multiplication" and $u : 1 \to M$ for

"unit", such that the following diagrams in $\mathcal{C}$ commute:

<div align="center">associativity   commutativity</div>

$$
\begin{array}{ccc}
M \times M \times M & \xrightarrow{\ \mathrm{id}_M \times m\ } & M \times M \\
{\scriptstyle m \times \mathrm{id}_M}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
M \times M & \xrightarrow[\ \ m\ \ ]{} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M \times M & \xrightarrow{\ \langle \pi_2, \pi_1 \rangle\ } & M \times M \\
{\scriptstyle m}\big\downarrow & \swarrow{\scriptstyle m} & \\
M & &
\end{array}
$$

<div align="center">unitality</div>

$$
M \xrightarrow{\ \mathrm{id}_M \times u\ } M \times M \xleftarrow{\ \eta \times u\ } M
$$
$$
{\scriptstyle \mathrm{id}_M} \searrow \quad \downarrow \quad \swarrow {\scriptstyle \mathrm{id}_M}
$$
$$
M
$$

This definition is *biased*, in the sense that it puts an emphasis on the operations $m : M \times M \to M$ and $u : 1 \to M$ rather than the other operations that exist in the commutative monoid M, such as the morphism

$$
M^5 \to M^2
$$
$$
(x_1, x_2, x_3, x_4, x_5) \mapsto (m(x_3, x_1), m(x_2, x_5)) \tag{6.5}
$$

The issue is that it is often unclear how to find homotopy-coherent variants for such biased definitions: for instance, the coherence conditions for symmetric monoidal categories (definition 5.2.2) are not evidently the right ones to consider, and we usually rely on highly non-trivial *coherence theorems* to reassure ourselves that the definitions we use are indeed well-behaved.

As we will now see, this issue usually disappears when we consider an unbiased definition. Generalizing the morphism (6.5), we see that every partial map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ induces a map

$$
M^m \to M^n
$$
$$
(x_i)_{1 \le i \le m} \mapsto \Big(\prod_{f(i)=j} x_i \Big)_{1 \le j \le n}
$$

where the $\prod$ notation indicates an iterated application of $m : M \times M \to M$.

Definition 6.10.1 ([Lur17, Notation 2.0.0.2]). Let $n \in \mathbb{N}$. Write $\langle n \rangle$ for the pointed set $\{*, 1, \ldots, n\}$. Write Fin_* for the category whose objects are the $\langle n \rangle$ for $n \in \mathbb{N}$ and the morphisms are maps $f : \langle m \rangle \to \langle n \rangle$ such that $f(*) = *$.

For every $1 \le i \le n$, let

$$
\rho^i(j) := \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise} \end{cases}
$$

Fact 6.10.2. *Every commutative monoid M in a cartesian category $\mathcal{C}$ induces a functor $F : \mathrm{Fin}_* \to \mathcal{C}$ such that:*

1. $F(\langle 1 \rangle) = M$

2. *the morphisms $F(\rho^i) : F(\langle n \rangle) \to F(\langle 1 \rangle)$ for $1 \le i \le n$ identify $F(\langle n \rangle)$ as an n-fold product of $F(1)$, so in particular*

$$
F(\langle n \rangle) \simeq M^n.
$$

This constructions induces an equivalence between the category CMon($\mathcal{C}$) *of commutative monoids in* $\mathcal{C}$ *and the full subcategory of* Fun(Fin$_*$, $\mathcal{C}$) *spanned by functors that satisfy item 2.*

Concretely, considering all the possible operations that exist in a commutative monoid allowed us to rephrase the very *definition* of commutative monoids in a more elementary way, and by elementary we mean in a way that refers to basic concepts of category theory: functors and products. Because of that, fact 6.10.2 can serve as the basis for the definition of commutative monoids in cartesian ∞-categories.

Definition 6.10.3. Let $\mathcal{C}$ be an ∞-category with finite products. A *commutative monoid* in $\mathcal{C}$ is a functor $F : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}$ such that the morphisms $F(\rho^i) : F(\langle n \rangle) \to F(\langle 1 \rangle)$ exhibit $F(\langle n \rangle)$ as an *n*-fold product of $F(\langle 1 \rangle)$ in $\mathcal{C}$ (i.e. as a limit of the discrete diagram consisting of *n* copies of $F(\langle 1 \rangle)$).

The category CMon$^\times$ ($\mathcal{C}$) of commutative monoids in $\mathcal{C}$ is defined to be the full subcategory of Fun (N (Fin$_*$), $\mathcal{C}$) on such functors. The evaluation at $\langle 1 \rangle$ determines a forgetful functor CMon$^\times$ ($\mathcal{C}$) $\to \mathcal{C}$.

Having defined commutative monoids, we could now simply define symmetric monoidal ∞-categories to be commutative monoids in ∞ Cat. However, for reasons that will soon become clear, it is more practical to work on the other side of the indexed/fibred equivalence (fact 6.4.27):

Definition 6.10.4 ([Lur17, Def 2.0.0.7])**.** A *symmetric monoidal* ∞-category is the data of an ∞-category $\mathcal{C}^\otimes$ together with a cocartesian fibration $p : \mathcal{C}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$, such that for every $n \in \mathbb{N}$, the functors $\rho^i_! : \mathcal{C}^\otimes_{\langle n \rangle} \to \mathcal{C}^\otimes_{\langle 1 \rangle}$ induce an equivalence of ∞-categories

$$\mathcal{C}^\otimes_{\langle n \rangle} \simeq (\mathcal{C}^\otimes_{\langle 1 \rangle})^n,$$

where $\mathcal{C}^\otimes_{\langle k \rangle} := p^{-1}(\langle k \rangle)$.

The ∞-category $\mathcal{C} := \mathcal{C}_{\langle 1 \rangle}$ is called the *underlying ∞-category* of $\mathcal{C}^\otimes$. We will sometimes abuse terminology and call the data of the cocartesian fibration $p : \mathcal{C}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ a *symmetric monoidal structure* on $\mathcal{C}$.

Note that by fact 6.2.4, this is indeed equivalent to commutative monoids in ∞ Cat. Moreover, identifying commutative monoid morphisms with morphisms in Fun (N (Fin$_*$), $\mathcal{C}$), the straightening/unstraightening equivalence also provides us with the following definition for symmetric monoidal functors:

Definition 6.10.5 ([Lur17, Def 2.1.3.7])**.** Let $p : \mathcal{C}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ and $q : \mathcal{D}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ be symmetric monoidal ∞-categories. A (strongly) *symmetric monoidal functor* is a functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ such that:

1. F maps p-cocartesian morphisms in $\mathcal{C}$ to q-cocartesian morphisms in $\mathcal{D}$,

2. the following triangle commutes in qCat:

$$
\begin{array}{ccc}
\mathcal{C}^\otimes & \xrightarrow{\quad F \quad} & \mathcal{D}^\otimes \\
& \searrow^{p} \quad \swarrow_{q} & \\
& \mathrm{N}\,(\mathrm{Fin}_*) &
\end{array}
$$

We write Fun$^\otimes$ ($\mathcal{C}^\otimes$, $\mathcal{D}^\otimes$) for the full subcategory of Fun$_{/\mathrm{N}\,(\mathrm{Fin}_*)}$ ($\mathcal{C}^\otimes$, $\mathcal{D}^\otimes$) (definition 6.4.30) spanned by the symmetric monoidal functors.

As expected, any symmetric monoidal ∞-category comes equipped with a unit and tensor product:

Definition 6.10.6. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*\,)$ be a symmetric monoidal ∞-category. Let $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ denoted the unique map, and $\beta : \langle 2 \rangle \to \langle 1 \rangle$ the map $1 \mapsto 1, 2 \mapsto 1, * \mapsto *$. α and β induce functors

$$\Delta^0 \xrightarrow{\ \simeq\ } \mathcal{C}^{\otimes}_{\langle 0 \rangle} \xrightarrow{\ \alpha_!\ } \mathcal{C}$$

$$\mathcal{C}^2 \xrightarrow{\ \simeq\ } \mathcal{C}^{\otimes}_{\langle 2 \rangle} \xrightarrow{\ \beta_!\ } \mathcal{C}$$

by covariant transport. The first functor determines an object $\mathbf{1} \in \mathcal{C}_0$ called the *unit* of the symmetric monoidal structure. The second functor is denoted

$$- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and is called the *tensor product* of the symmetric monoidal structure.

Remark 6.10.7. One can check by unfolding definitions that $\mathbf{1}$ and $\otimes$ satisfy the usual axioms of symmetric monoidal structures up to homotopy. For instance, the map $\gamma : \langle 3 \rangle \to \langle 1 \rangle$ given by $1, 2, 3 \mapsto 1$ induces a functor

$$(- \otimes - \otimes -) : \mathcal{C}^3 \simeq \mathcal{C}^{\otimes}_{\langle 3 \rangle} \to \mathcal{C},$$

and the commutativity of the following diagram in Fin_* :

$$
\begin{array}{ccc}
& \overset{\substack{1,2 \mapsto 1 \\ 3 \mapsto 2}}{\longrightarrow} & \\
\langle 3 \rangle & & \langle 2 \rangle \\
{\scriptstyle \substack{1 \mapsto 1 \\ 2,3 \mapsto 2}} \Big\downarrow & \overset{\gamma}{\searrow} & \Big\downarrow {\scriptstyle \beta} \\
\langle 2 \rangle & \underset{\beta}{\longrightarrow} & \langle 1 \rangle
\end{array}
$$

induces a homotopy commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{(-\otimes-)\times \mathrm{id}_{\mathcal{C}}} & \mathcal{C} \times \mathcal{C} \\
{\scriptstyle \mathrm{id}_{\mathcal{C}} \times (-\otimes-)} \Big\downarrow & {\scriptstyle (-\otimes-\otimes-)} \searrow & \Big\downarrow {\scriptstyle -\otimes-} \\
\mathcal{C} \times \mathcal{C} & \underset{-\otimes-}{\longrightarrow} & \mathcal{C}
\end{array}
$$

meaning there are isomorphisms

$$a \otimes (b \otimes c) \simeq a \otimes b \otimes c \simeq (a \otimes b) \otimes c$$

natural in a, b and c, witnessing the associativity of the tensor product up to homotopy.

Remark 6.10.8. The definition of the unit $\mathbf{1}$ and of the tensor product $\otimes$ depends on the choice of an inverse to the equivalence $\mathcal{C}^{\otimes}_{\langle k \rangle} \simeq \mathcal{C}^k$, so it is technically only well-defined up to a contractible space of choices, but as we have seen by now, this is to be expected when working homotopy-theoretically.

Fact 6.10.9 ([Lur17, Rmk 2.1.3.8])**.** *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories. Then F is an equivalence if and only if its restriction to the underlying ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence, in which case we say it is a* symmetric monoidal equivalence*.*

We now review a few examples of symmetric monoidal structures on ∞-categories.

Example 6.10.10 ([Lur17, Ex 2.1.2.21, Constr 2.1.1.7])**.** Let $\mathcal{C}$ be a symmetric monoidal category. Then one can define a category $\mathcal{C}^{\otimes}$ whose objects are tuples $(c_1, \ldots, c_n)$ of objects of $\mathcal{C}$ with $n \geq 0$, and whose morphisms

$$(c_1, \ldots, c_m) \to (d_1, \ldots, d_n)$$

are given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ in Fin_* together with, for every $1 \leq j \leq n$, a morphism

$$\bigotimes_{i \in \alpha^{-1}(j)} c_i \to d_j$$

in $\mathcal{C}$. The composition of morphisms is given by the monoidal structure on $\mathcal{C}$.

There is an evident forgetful functor $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$, which turns out to be a Grothendieck opfibration. Applying the nerve, we thus get a cocartesian fibration $\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ satisfying the additional condition of definition 6.10.4, so $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, whose underlying ∞-category is canonically isomorphic to $\mathrm{N}(\mathcal{C})$.

In this way, every symmetric monoidal category can be seen as a symmetric monoidal ∞-category, and reciprocally any symmetric monoidal structure on the ∞-category $\mathrm{N}(\mathcal{C})$ determines a symmetric monoidal structure on $\mathcal{C}$.

Remark 6.10.11*.* The identity functor $\mathrm{N}(\mathrm{Fin}_*) \to \mathrm{N}(\mathrm{Fin}_*)$ identifies $\mathrm{N}(\mathrm{Fin}_*)$ as a symmetric monoidal ∞-category, whose underlying ∞-category is isomorphic to Δ^0: this is the unique symmetric monoidal structure on the terminal ∞-category. Note that this can also be obtained using the nerve construction (example 6.10.10) applied to the unique symmetric monoidal structure on the terminal category.

Example 6.10.12. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ a symmetric monoidal ∞-category, corresponding to a functor $F : \mathrm{N}(\mathrm{Fin}_*) \to \infty\mathrm{Cat}$. Then, post-composing with the equivalence $\mathrm{op} : \infty\mathrm{Cat} \simeq \infty\mathrm{Cat}$ yields a functor

$$\mathrm{N}(\mathrm{Fin}_*) \xrightarrow{\ F\ } \infty\mathrm{Cat} \xrightarrow{\ \mathrm{op}\ } \infty\mathrm{Cat}$$

corresponding to a symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$ called the *opposite symmetric monoidal structure* and written $(\mathcal{C}^{\mathrm{op}})^{\otimes}$, with tensor product given by

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} = (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \xrightarrow{\otimes^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$$

Since op is an equivalence, this moreover means that the data of a symmetric monoidal structure on $\mathcal{C}$ is equivalent to the data of a symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$.

As one can expect, any ∞-category with finite products admits a canonical symmetric monoidal structure given by its terminal object and cartesian products. This can be made formal through the notion of *cartesian symmetric monoidal structure*:

Definition 6.10.13 ([Lur17, Def 2.4.0.1])**.** Let $\mathcal{C}$ be an ∞-category. A symmetric monoidal structure on $\mathcal{C}$ is said to be *cartesian* if:

1. the unit is a terminal object in $\mathcal{C}$,

2. for every pair of objects $x, y \in \mathcal{C}_0$, the morphisms

$$x \otimes y \to x \otimes \mathbf{1} \simeq x$$
$$x \otimes y \to \mathbf{1} \otimes y \simeq y$$

induced by the terminality of $\mathbf{1}$ exhibit $x \otimes y$ as a product of x and y in $\mathcal{C}$.

Dually, it is said to be *cocartesian* if:

1. the unit is an initial object in $\mathcal{C}$,

2. for every pair of objects $x, y \in \mathcal{C}_0$, the morphisms

$$x \simeq x \otimes \mathbf{1} \to x \otimes y$$
$$y \simeq \mathbf{1} \otimes y \to x \otimes y$$

induced by the initiality of $\mathbf{1}$ exhibit $x \otimes y$ as a coproduct of x and y in $\mathcal{C}$.

Remark 6.10.14. A symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is cartesian symmetric monoidal if and only if the opposite symmetric monoidal structure $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ is cocartesian.

It can be shown through an explicit construction that such structures always exist and are essentially unique up to homotopy:

Fact 6.10.15 ([Lur17, Prop 2.4.1.5, Cor 2.4.1.8]). *Let $\mathcal{C}$ be an ∞-category with finite products. Then there exists a cartesian symmetric monoidal ∞-category $\mathcal{C}^{\times}$ such that $\mathcal{C}^{\times}_{\langle 1 \rangle} \simeq \mathcal{C}$.*

Moreover, if $\mathcal{C}^{\otimes}$ is a cartesian symmetric monoidal ∞-category such that $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$, then there is a symmetric monoidal equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{\times}$ whose restriction to $\mathcal{C} \to \mathcal{C}$ is naturally isomorphic to the identity functor.

We can now link the definition of symmetric monoidal ∞-category to slogan 6.9.1: starting from an isofibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$, what is really needed to state that there are canonical equivalences $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$ is all the morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ admits cocartesian lifts with respect to to p. In that case, given objects $x_1, \dots, x_n$ and y in $\mathcal{C}$, one can define the ∞-groupoid

$$\mathrm{Hom}_{\mathcal{C}}(x_1, x \dots, x_n \,;\, y) := \mathrm{Hom}_{\mathcal{C}^{\otimes}}((x_1, \dots, x_n), y),$$

where $(x_1, \dots, x_n)$ is an object of $\mathcal{C}^{\otimes}_{\langle n \rangle}$ obtained through the equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n$.

Also, the covariant transport functors $\rho^i_! : \mathcal{C}^n \to \mathcal{C}$ simply encode projections, and not any meaningful information about the tensor product or the symmetric monoidal structure. More generally, this is true of the following class of morphisms in Fin_*:

Definition 6.10.16 ([Lur17, Def 2.1.1.8]). A morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_* is said to be *inert* if, for every element $i \in \langle n \rangle \setminus \{*\}$, the inverse image $f^{-1}(i)$ is a singleton.

Indeed, for every inert morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_*, the covariant transport functor $f_! : \mathcal{C}^m \to \mathcal{C}^n$ only ever encodes a composition of projections and permutations. This motivates the following definition:

Definition 6.10.17 ([Lur17, Def 2.1.1.10, Rmk 2.1.1.14]). An ∞-*operad* (or ∞-*multicategory*) is an isofibration $p : \mathcal{O}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ of ∞-categories such that:

1. every inert morphism $f : \langle m \rangle \to \langle n \rangle$ ins $\mathrm{N}\,(\mathrm{Fin}_*)$ admits cocartesian lift with respect to p,

2. the induced covariant transport functors $\rho^i_{\,!} : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ induce an equivalence

$$\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^n.$$

Definition 6.10.18. The cocartesian lifts of inert morphisms in $\mathcal{O}^{\otimes}$ are called *inert morphisms* in $\mathcal{O}^{\otimes}$.

Remark 6.10.19. In particular, a symmetric monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ is precisely an ∞-operad that admits cocartesian lifts for *every* morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_*, and not just the inert ones. This respects the intuition of slogan 6.9.1 that being a symmetric monoidal ∞-category should be a *property* of ∞-operads/∞-multicategories.

Example 6.10.20. Let $\mathrm{Triv}^{\otimes}$ denote the subcategory of $\mathrm{N}\,(\mathrm{Fin}_*)$ spanned all the objects and only the inert morphisms. Then the inclusion $p : \mathrm{Triv}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ is an ∞-operad called the *trivial ∞-operad*, and not a symmetric monoidal ∞-category.

We will see other examples of ∞-operads that are not symmetric monoidal ∞-categories, and their usefulness, in section 6.13.

6.11   Commutative monoids

The next logical step in the study of symmetric monoidal ∞-categories is the definition of commutative monoids. Given a symmetric monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$, a commutative monoid in $\mathcal{C}^{\otimes}$ should in particular have an underlying object x in $\mathcal{C}$, and a multiplication morphism $x \otimes x \to x$, as well as a unit morphism $\mathbf{1} \to x$. This can be represented instead as morphisms $(x, x) \to x$ and $() \to x$ in $\mathcal{C}^{\otimes}$. More generally, any morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_* should induce a morphism $x^{\otimes m} \to x^{\otimes n}$, in turn represented as morphism in $\mathcal{C}^{\otimes}$. As we can see, the fact that morphisms $(x, x) \to x$ in $\mathcal{C}^{\otimes}$ can be represented as morphisms $x \otimes x \to x$ in $\mathcal{C}$ is not even required to make sense of the notion of commutative monoid, which leads us to the following definition:

Definition 6.11.1. A *commutative monoid* in an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ is a functor $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ such that:

1. $p \circ M = \mathrm{id}_{\mathrm{N}\,(\mathrm{Fin}_*)}$,

2. M maps inert morphisms in Fin_* to inert morphisms in $\mathcal{C}^{\otimes}$ (in the sense of definition 6.10.18).

The ∞-category of commutative monoids in $\mathcal{O}^{\times}$, denoted $\mathrm{CMon}\,(\mathcal{O}^{\otimes})$ is by definition the full subcategory of $\mathrm{Fun}_{/\,\mathrm{N}\,(\mathrm{Fin}_*)}\,(\mathrm{N}\,(\mathrm{Fin}_*), \mathcal{O}^{\otimes})$ spanned by the functors satisfying condition 2.

The *underlying object* of a commutative monoid $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ is the object

$$|M| := M(\langle 1 \rangle)$$

Given a commutative monoid $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{O}^{\otimes}$ in an ∞-operad $p : \mathcal{O} \to \mathrm{N}\,(\mathrm{Fin}_*)$, the condition 2 in definition 6.11.1 means in particular that the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in Fin_* allow us to identify $M(\langle n \rangle)$ with the tuple $(M(\langle 1 \rangle), \dots, M(\langle 1 \rangle))$ in $\mathcal{O}^{\otimes}$.

Remark 6.11.2. In *Higher Algebra* [Lur17], Lurie refers to commutative monoids as *commutative algebras*, reserving the term of commutative monoid for commutative algebras with respect to cartesian symmetric monoidal structures. We will stick with using the term *commutative monoid* since it is more common in the Linear Logic community to speak of comonoids than of coalgebras.

Remark 6.11.3 ([Lur17, Prop 2.4.2.5]). Given an ∞-category $\mathcal{C}$ with finite products, there are a priori two conflicting notions of commutative monoids in $\mathcal{C}$:

1. the commutative monoids defined as functors $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}$ such that $M(\langle n \rangle) \simeq (M(\langle 1 \rangle))^n$ (definition 6.10.3),

2. the commutative monoids in the cartesian symmetric monoidal ∞-category $\mathcal{C}^\times$, i.e. functors $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}^\times$ preserving inert morphisms (definition 6.11.1).

As it turns out, these notions coincide: there is a canonical equivalence

$$\mathrm{CMon}\,(\mathcal{C}^\times) \simeq \mathrm{CMon}^\times(\mathcal{C}).$$

Just like in ordinary category theory, every object in an ∞-category with finite coproducts admits a unique commutative monoid structure with respect to coproducts:

Fact 6.11.4 ([Lur17, Cor 2.4.3.10]). *If $\mathcal{C}^\otimes$ is a cocartesian symmetric monoidal ∞-category, the forgetful functor $\mathrm{CMon}\,(\mathcal{C}^\otimes) \to \mathcal{C}$ is an equivalence of ∞-categories. In other words, every object of $\mathcal{C}$ admits a unique commutative monoid structure with respect to the coproduct in $\mathcal{C}$, up to a contractible space of choices. Intuitively, this is because the unit $\mathbf{1} \to |M|$ of the commutative monoid must be the unique (up to homotopy) morphism from the initial object $\mathbf{1}$ to X, and the associativity and unitality conditions enforce the multiplication morphism $|M| \sqcup |M| \to |M|$ to be given by $[\mathrm{id}, \mathrm{id}]$.*

In ordinary algebra, coproducts in the category of rings (or k-algebras) are computed as the tensor product of their underlying abelian groups. This idea can be vastly generalized:

Fact 6.11.5 ([Lur17, Ex 3.2.4.4, Prop 3.2.4.7]). *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Then $\mathrm{CMon}\,(\mathcal{C}^\otimes)$ admits finite coproducts, and the forgetful functor $\mathrm{CMon}\,(\mathcal{C}^\otimes) \to \mathcal{C}$ extends to a symmetric monoidal functor*

$$\mathrm{CMon}\,(\mathcal{C}^\otimes)^\sqcup \to \mathcal{C}^\otimes,$$

(where $\mathrm{CMon}\,(\mathcal{C}^\otimes)^\sqcup$ denotes the cocartesian symmetric monoidal structure on $\mathrm{CMon}\,(\mathcal{C}^\otimes)$). In other words, $\mathrm{CMon}\,(\mathcal{C}^\otimes)$ admits finite coproducts, and they are computed as the tensor product and unit in $\mathcal{C}^\otimes$.

6.11.1 Lax symmetric monoidal functors

Generalizing upon the idea of definition 6.11.1, we can define lax monoidal functors:

Definition 6.11.6. Let $p : \mathcal{O}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ and $q : \mathcal{O}'^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ be ∞-operads. An ∞-*operad map* from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$ is a functor $F : \mathcal{O}^\otimes \to \mathcal{O}'^\otimes$ such that:

1. $q \circ F = p$,

2. F maps inert morphisms in $\mathcal{O}^\otimes$ to inert morphisms in $\mathcal{O}'^\otimes$.

When $p : \mathcal{C}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ and $q : \mathcal{D}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ are symmetric monoidal ∞-categories, ∞-operad maps from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$ are also called *lax symmetric monoidal* functors. The ∞-category of lax symmetric monoidal functors from $\mathcal{C}^\otimes$ to $\mathcal{D}^\otimes$, denoted $\mathrm{Fun}^{\mathrm{lax}}\,(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$, is by definition the full subcategory of $\mathrm{Fun}_{/\,\mathrm{N}\,(\mathrm{Fin}_*)}\,(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned the lax symmetric monoidal functors.

Remark 6.11.7. Note that by definition we have that $\mathrm{Fun}^\otimes\,(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \subseteq \mathrm{Fun}^{\mathrm{lax}}\,(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ is a full subcategory. More generally, the objects of $\mathrm{Fun}^\otimes\,(\mathcal{O}^\otimes, \mathcal{D}^\otimes)$ can be thought of as $\mathcal{O}$-*algebras* in $\mathcal{D}$ (so much so that Lurie uses the notation $\mathrm{Alg}_\mathcal{O}(\mathcal{D})$ for $\mathrm{Fun}^{\mathrm{lax}}\,(\mathcal{O}^\otimes, \mathcal{D}^\otimes)$ in [Lur17]). More generally, much of the theory of symmetric monoidal ∞-categories is developed in [Lur17] relative to an arbitrary ∞-operad $\mathcal{O}$ instead of $\mathrm{N}\,(\mathrm{Fin}_*)$.

Example 6.11.8. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the forgetful functor $\mathrm{Fun}^{\mathrm{lax}}\left(\mathrm{Triv}^{\otimes}, \mathcal{C}^{\otimes}\right) \to \mathcal{C}$ is an equivalence of ∞-categories. Following the intuition of remark 6.11.7, this means that the "algebras" in $\mathcal{C}^{\otimes}$ for the trivial ∞-operad are just the objects of $\mathcal{C}$, hence the name of trivial ∞-operad.

Example 6.11.9. Recall from remark 6.10.11 that the identity functor $\mathrm{id} : \mathrm{N}\left(\mathrm{Fin}_*\right) \to \mathrm{N}\left(\mathrm{Fin}_*\right)$ can be seen as exhibiting the unique symmetric monoidal structure on the terminal ∞-category Δ^0. Given a symmetric monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}\left(\mathrm{Fin}_*\right)$, we have

$$\mathrm{Fun}^{\mathrm{lax}}\left(\mathrm{N}\left(\mathrm{Fin}_*\right), \mathcal{C}^{\otimes}\right) = \mathrm{CMon}\left(\mathcal{O}^{\otimes}\right)$$

simply from unfolding definitions. This corresponds to the standard fact from ordinary category theory that commutative monoids correspond to lax symmetric monoidal functors from the terminal category.

To see that definition 6.11.6 is a reasonable definition for lax symmetric monoidal functors, consider such a functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. Given objects $x, y \in \mathcal{C}_0$, the preservation of the cocartesian lifts for $\rho^1, \rho^2 : \langle 2 \rangle \to \langle 1 \rangle$ means in particular that

$$F((x, y)) \simeq (F(x), F(y))$$

under the identifications $\mathcal{C}_{\langle 2 \rangle}^{\otimes} \simeq \mathcal{C}^2$ and $\mathcal{D}_{\langle 2 \rangle}^{\otimes} \simeq \mathcal{D}^2$. In particular, the canonical morphism $(x, y) \to x \otimes y$ in $\mathcal{C}^{\otimes}$ is mapped by F to a morphism $(F(x), F(y)) \to F(x \otimes y)$ in $\mathcal{D}$, thus inducing a morphism

$$F(x) \otimes F(y) \to F(x \otimes y)$$

in $\mathcal{D}$ witnessing lax monoidality in the ordinary sense. A similar argument can be used to obtain a morphism $\mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$.

Note that F being (strongly) symmetric monoidal precisely implies that the morphisms $(F(x), F(y)) \to F(x \otimes y)$ in $\mathcal{D}$ are also q-cocartesian, so that the morphisms $F(x) \otimes F(y) \to F(x \otimes y)$ are isomorphisms, which is precisely the kind of condition under which lax monoidal functors are considered to be strongly symmetric monoidal.

Remark 6.11.10. It is clear from the definition that lax symmetric monoidal functors compose. In particular, using example 6.11.9, use see that any lax symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ induces a functor $\mathrm{CMon}\left(F\right) : \mathrm{CMon}\left(\mathcal{C}^{\otimes}\right) \to \mathrm{CMon}\left(\mathcal{D}^{\otimes}\right)$.

When considering functors between symmetric monoidal ∞-categories, we can also consider oplax symmetric monoidal functors:

Definition 6.11.11. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\left(\mathrm{Fin}_*\right), q : \mathcal{D}^{\otimes} \to \mathrm{N}\left(\mathrm{Fin}_*\right)$ be symmetric monoidal ∞-categories. An *oplax symmetric monoidal* functor from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is a lax symmetric monoidal functor from $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ to $(\mathcal{D}^{\mathrm{op}})^{\otimes}$. The ∞-category of oplax symmetric monoidal functors from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is defined to be

$$\mathrm{Fun}^{\mathrm{opl}}\left(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}\right) := \left(\mathrm{Fun}^{\mathrm{lax}}\left((\mathcal{C}^{\mathrm{op}})^{\otimes}, (\mathcal{D}^{\mathrm{op}}), \otimes\right)\right)^{\mathrm{op}}.$$

Remark 6.11.12. Note that while lax symmetric monoidal functors make sense between arbitrary ∞-categories, for oplax ones we need symmetric monoidal structures: this is because the definition of ∞-operad is not self-dual. Indeed, while the data of a symmetric monoidal structure on $\mathcal{C}$ or $\mathcal{C}^{\mathrm{op}}$ is equivalent (example 6.10.12), and "∞-operad structure" on an ∞-category $\mathcal{C}$ would correspond to something like an "∞-co-operad structure" on $\mathcal{C}^{\mathrm{op}}$: while operads have morphisms with multiple inputs and one output, the morphisms in a cooperad have a single input and multiple outputs.

In the 1-categorical theory of symmetric monoidal structures, it is well-known that left adjoints to lax monoidal functors are oplax monoidal, and dually right adjoints to oplax monoidal functors are lax monoidal. The same is true in the ∞-categorical setting:

Fact 6.11.13 ([Lur09, Cor 7.3.2.7])**.** *Let $p : \mathcal{C}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$, $q : \mathcal{D}^\otimes \to \mathrm{N}\,(\mathrm{Fin}_*)$ be symmetric monoidal ∞-categories.*

1. *Let $R : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ be a lax symmetric monoidal functor whose restriction to $\mathcal{D} \to \mathcal{C}$ admits a left adjoint. Then R admits a left adjoint $L : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ which is moreover oplax symmetric monoidal.*

2. *Let $L : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be an oplax symmetric monoidal functor whose restriction to $\mathcal{C} \to \mathcal{D}$ admits a right adjoint. Then L admits a right adjoint $R : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ which is moreover lax symmetric monoidal.*

Remark 6.11.14*.* This result has been greatly enhanced to a contravariant equivalence of $(\infty, 2)$-categories between

- symmetric monoidal ∞-categories, lax monoidal functors and monoidal natural transformations,

- symmetric monoidal ∞-categories, oplax monoidal functors and monoidal natural transformations,

by Haugseng, Hebestreit, Linskens and Nuiten in [Hau+23].

6.11.2 Free commutative monoids

As we will be interested in cofree commutative comonoids to define Lafont ∞-categories in section 7.1, we now deal with the more classical theory of free commutative monoids in the ∞-categorical setting.

Definition 6.11.15 ([Lur09, Prop 3.1.3.2])**.** Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category, and X be an object of $\mathcal{C}$. A *free commutative monoid* on X is a commutative monoid $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}^\otimes$ together with a morphism $f : X \to |M|$ in $\mathcal{C}$ such that for every commutative monoid N in $\mathcal{C}^\otimes$, the map

$$\mathrm{Hom}_{\mathrm{CMon}\,(\mathcal{C}^\otimes)}(M, N) \xrightarrow{\;|-|\;} \mathrm{Hom}_{\mathcal{C}}(|M|, |N|) \xrightarrow{\;-\circ f\;} \mathrm{Hom}_{\mathcal{C}}(X, |N|)$$

induced by precomposition with f is a homotopy equivalence.

Remark 6.11.16*.* Definition 6.11.15 can be rephrased as saying that a free commutative monoid on X is an object $M \in \mathrm{CMon}\,(\mathcal{C}^\otimes)_0$ that corepresents the copresheaf

$$\begin{aligned} \mathrm{CMon}\,(\mathcal{C}^\otimes) &\to \mathcal{S} \\ N &\mapsto \mathrm{Hom}_{\mathcal{C}}(X, |N|) \end{aligned}$$

on $\mathrm{CMon}\,(\mathcal{C}^\otimes)$. Hence, by fact 6.5.14 the space of free commutative monoids on X is contractible if non-empty, so that we may speak of *the* free commutative monoid on X.

Remark 6.11.17*.* Suppose that for every X in $\mathcal{C}$, there exists a free commutative monoid on X. Then by fact 6.7.12, the forgetful functor $|-| : \mathrm{CMon}\,(\mathcal{C}^\otimes) \to \mathcal{C}$ admits a left adjoint $F : \mathcal{C} \to \mathrm{CMon}\,(\mathcal{C}^\otimes)$, and for every $X \in \mathcal{C}_0$ the unit morphism $\eta_X : X \to |F(X)|$ exhibits $F(X)$ as a free commutative monoid on X.

In ordinary category theory, when a symmetric monoidal category $\mathcal{C}$ admits enough colimits, and if those colimits are compatible with the tensor product of $\mathcal{C}$, then $\mathcal{C}$ can be shown to admit free commutative monoids, and they can be computed explicitly via the formula

$$X \mapsto \bigsqcup_{n \geq 0} X^n / \mathfrak{S}_n.$$

The same result is true in the ∞-categorical setting, replacing quotients by homotopy quotients:

Definition 6.11.18 ([Lur17, Construction 3.1.3.9]). Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category, and X an object of $\mathcal{C}$. By example 6.11.8, X determines a lax symmetric monoidal functor $\overline{X} : \mathrm{Triv}^{\otimes} \to \mathcal{C}$. Note that the canonical functor $\mathbf{B}\mathfrak{S}_n \to \mathrm{N}\,(\mathrm{Fin}_*)$ factors through $\mathrm{Triv}^{\otimes}$ because isomorphisms in $\mathrm{N}\,(\mathrm{Fin}_*)$ are inert morphisms, so we get a composite diagram

$$\theta_n : \mathbf{B}\mathfrak{S}_n \to \mathrm{Triv}^{\otimes} \xrightarrow{\overline{X}} \mathcal{C}^{\otimes}.$$

Let $\lambda_n : \langle n \rangle \to \langle 1 \rangle$ denote the morphism given by $1, \ldots, n \mapsto 1$. Since $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ is a cocartesian fibration, we get a diagram

$$\lambda_{n!} \circ f : \mathfrak{S}_n \to \mathcal{C}$$

such that $\lambda_{n!}(\theta_n(*)) = x \otimes \cdots \otimes x$ is the n-fold tensor product of x with itself.

This is the *canonical action of $\mathfrak{S}_n$ on $X^{\otimes n}$*. We write $X^{\otimes n} /\!\!/ \mathfrak{S}_n$ for the colimit (if it exists) of the diagram

$$\lambda_{n!} \circ \theta_n : \mathbf{B}\mathfrak{S}_n \to \mathcal{C}.$$

We write

$$\mathrm{Sym}\,(X) := \bigsqcup_{n \geq 0} X^{\otimes n} /\!\!/ \mathfrak{S}_n,$$

i.e. the colimit of the diagram

$$\bigsqcup_{n \geq 0} \mathbf{B}\mathfrak{S}_n \xrightarrow{[\lambda_{n!} \circ \theta_n]_{n \geq 0}} \mathcal{C}$$

if such a colimit exists.

Fact 6.11.19 ([Lur17, Prop 3.1.3.3, Prop 3.1.3.13]). *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and X an object of $\mathcal{C}$. Suppose that the colimit $\bigsqcup_{n \geq 0} X^{\otimes n} /\!\!/ \mathfrak{S}_n$ exists and that, for every object $A \in \mathcal{C}_0$, the canonical morphisms*

$$X^{\otimes n} \otimes A \to \mathrm{Sym}\,(X) \otimes A$$

exhibit $\mathrm{Sym}\,(X) \otimes A$ *as a colimit*

$$\mathrm{Sym}\,(X) \otimes A \simeq \left(\bigsqcup_{n \geq 0} (X^{\otimes n} \otimes A) /\!\!/ \mathfrak{S}_n \right).$$

Then:

1. *there is a free commutative monoid on X in $\mathcal{C}$ whose underlying object in $\mathcal{C}$ is $\mathrm{Sym}\,(X)$,*

2. *for every commutative monoid $M : \mathrm{N}\,(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ in $\mathcal{C}$, a morphism $f : X \to |M|$ exhibits M as a free commutative monoid over X if and only if for every $n \geq 0$, the canonical morphisms*

$$X^{\otimes} /\!\!/ \mathfrak{S}_n \to |M|$$

induced by the cocartesianness of $\lambda_n : \langle n \rangle \to \langle 1 \rangle$ exhibit $M(\langle 1 \rangle)$ as a colimit

$$\bigsqcup_{n \geq 0} X^{\otimes n} /\!\!/ \mathfrak{S}_n \simeq |M|.$$

As a corollary, we have:

Fact 6.11.20 ([Lur17, Ex 3.1.3.14])**.** *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category whose underlying ∞-category $\mathcal{C}$ admits countable colimits (i.e. colimits of diagrams indexed by simplicial sets with a countable number of simplices), and suppose that for every $X \in \mathcal{C}_0$ the tensor product functor $Y \mapsto X \otimes Y$ preserves countable colimits. Then every object $X \in \mathcal{C}_0$ admits a free commutative monoid, given by $\mathrm{Sym}\,(X)$.*

Example 6.11.21. The ∞-categories $\mathcal{S}$ and $\infty\,\mathrm{Cat}$ are cartesian closed, i.e. the functors

$$X \times - : \mathcal{S} \to \mathcal{S} \quad \text{and} \quad \mathcal{C} \times - : \infty\,\mathrm{Cat} \to \infty\,\mathrm{Cat}$$

admit right adjoints, so by fact 6.7.18, the tensor products of the cartesian structures on $\mathcal{S}$ and $\infty\,\mathrm{Cat}$ preserve colimits independently in both variables, so they satisfy the hypothesis of fact 6.11.20. Thus $\mathcal{S}$ and $\infty\,\mathrm{Cat}$ admit free commutative monoids on every object. In particular, this means we can build the *free symmetric monoidal ∞-category* $\mathrm{Sym}\,(\mathcal{C})$ on any ∞-category $\mathcal{C}$, and it is moreover defined by the formula

$$\mathrm{Sym}\,(\mathcal{C}) := \bigsqcup_{n \geq 0} \mathcal{C}^{\times n} \mathbin{/\!\!/} \mathfrak{S}_n,$$

thus generalizing the standard construction of the free symmetric monoidal category on a category.

Remark 6.11.22. Using the explicit formula for homotopy quotients given in fact 6.6.29, and since coproducts in $\infty\,\mathrm{Cat}$ are computed as coproducts in sSet, we can explicitly define $\mathrm{Sym}\,(\mathcal{C})$ to be the simplicial set

$$\mathrm{Sym}\,(\mathcal{C}) := \bigsqcup_{n \geq 0} (\mathbf{E}\mathfrak{S}_n \times \mathcal{C}^{\times n}) / \mathfrak{S}_n.$$

In particular, when $\mathcal{C} := \mathrm{N}\,C$ is the nerve of an ordinary category C, we recover the usual description for the free symmetric monoidal category on C, whose objects are lists $(x_1, \dots, x_m)$ of elements of C, and whose morphisms $f : (x_1, \dots, x_m) \to (y_1, \dots, y_n)$ exist only when $m = n$ and are given by a permutation $\sigma \in \mathfrak{S}_n$ together with a family of morphisms $f_i : x_i \to y_{\sigma(i)}$.

Remark 6.11.23. The ∞-category $\bigsqcup_{n \geq 0} \mathbf{B}\mathfrak{S}_n$ is the nerve of a groupoid that's equivalent to the groupoid Fin of finite sets and bijections. Combining this with example 6.11.21 and remark 6.6.25, we see that the free commutative monoid on an ∞-groupoid X corresponds in homotopy type theory to the type

$$\mathrm{HMul}\,(X) := \sum_{E : \mathrm{Fin}} (E \to X),$$

of homotopy multisets on X (definition 5.7.6). This means that in the same way that multisets on a set X give the free commutative monoid on X in Set, homotopy multisets on an ∞-groupoid (or type) X indeed correspond to the free commutative monoid on X in an ∞-categorical sense, thus giving another justification for definition 5.7.6.

6.11.3 Limits and colimits of commutative monoids

We now turn to the computation of limits and colimits in $\mathrm{CMon}\,(\mathcal{C}^{\otimes})$. Of particular interest is fact 6.11.26 about the presentability of ∞-categories of commutative monoids, a result we will adapt in theorem 7.1.15 to construct cofree commutative comonoids and thus construct free exponentials of Linear Logic in the ∞-categorical context.

Fact 6.11.24 ([Lur17, Cor 3.2.2.5])**.** *Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category and K a simplicial set. Suppose that the underlying ∞-category $\mathcal{C}$ of $\mathcal{C}^{\otimes}$ admits K-indexed limits. Then:*

1. CMon $(\mathcal{C}^{\otimes})$ *admits K-indexed limits,*

2. *given a diagram $u : K \to$ CMon $(\mathcal{C}^{\otimes})$, a commutative monoid M in $\mathcal{C}^{\otimes}$ and a natural transformation $\alpha : \underline{M} \Rightarrow u$, α exhibits M as a limit of u if and only if $|\alpha| : |\underline{M}| \Rightarrow |u|$ exhibits $|M|$ as a limit of $|u|$ in $\mathcal{C}$.*

In particular, the forgetful functor CMon $(\mathcal{C}^{\otimes}) \to \mathcal{C}$ *preserves K-indexed limits.*

The situation for colimits is a bit more subtle, as colimits in CMon $(\mathcal{C}^{\otimes})$ are generally not computed as in $\mathcal{C}$, and we also need the tensor product to be compatible with colimits:

Fact 6.11.25 ([Lur17]Cor 3.2.3.2-3)**.** *Let $p : \mathcal{C}^{\otimes} \to$ N (Fin$_*$) be a symmetric monoidal ∞-category and κ an uncountable regular cardinal. Suppose that $\mathcal{C}^{\otimes}$ admits κ-small colimits and that for every $X \in \mathcal{C}_0$, the tensor product functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ preserves κ-small colimits.*
Then CMon $(\mathcal{C}^{\otimes})$ *admits κ-small colimits. Moreover, the forgetful functor* CMon $(\mathcal{C}^{\otimes}) \to \mathcal{C}$ *preserves K-indexed colimits for every* sifted *simplicial set K.*

In particular, if $\mathcal{C}^{\otimes}$ is such that $\mathcal{C}$ is a presentable ∞-category and that $X \otimes - : \mathcal{C} \to \mathcal{C}$ preserves small colimits for every $X \in \mathcal{C}_0$, then CMon $(\mathcal{C}^{\otimes})$ admits small colimits. It turns out that this is also sufficient for CMon $(\mathcal{C}^{\otimes})$ to be presentable:

Fact 6.11.26 ([Lur17]Cor 3.2.3.5)**.** *If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category such that*

- $\mathcal{C}$ *is accessible,*

- *there exists κ such that for every $X \in \mathcal{C}_0$, $X \otimes - : \mathcal{C} \to \mathcal{C}$ preserves κ-filtered colimits,*

then CMon $(\mathcal{C}^{\otimes})$ *is accessible.*
As a corollary of this together with fact 6.11.25, if $\mathcal{C}$ is moreover presentable then so is CMon $(\mathcal{C}^{\otimes})$.

Proof idea. We briefly mention the main idea of the proof, since we will use the same principle again in the proof of theorem 7.1.15. First recall that CMon $(\mathcal{C}^{\otimes})$ is defined to be the full subcategory of Fun$_{/ \text{N} (\text{Fin}_*)}$ (N (Fin$_*$), $\mathcal{C}^{\otimes}$) spanned by the functors that map inert morphisms in N (Fin$_*$) to inert morphisms in $\mathcal{C}^{\otimes}$. Rather than work with this specific ∞-category, Lurie answers the more general question:

"Given a subcategory $\mathbb{C} \subseteq \infty \text{Cat}$, a cocartesian fibration $p : X \to S$ classifying a functor $S \to \mathbb{C}$ and a class E of edges in S, under which conditions on $\mathbb{C}$ does the ∞-category Fun$_{/S}{}^E (S, X)$
belong to $\mathbb{C}$?"

where Fun$_{/S}{}^E (S, X)$ denotes the full subcategory of Fun$_{/S}$ (S, X) spanned by the functors that map every morphism in E to a cocartesian lift.
As Lurie shows, the following three conditions on $\mathbb{C}$ suffice:

1. $\mathbb{C}$ is admits small limits, and the inclusion $\mathbb{C} \subseteq \infty \text{Cat}$ preserves small limits,

2. if $X \in \mathbb{C}_0$, then Fun $(\Delta^1, X) \in \mathbb{C}_0$,

3. given $X, Y \in \mathbb{C}_0$, a functor $F : X \to$ Fun (Δ^1, Y) is a morphism in $\mathbb{C}$ if and only if the two induced functors $F_0, F_1 : X \to Y$ are morphisms in $\mathbb{C}$.

This can then be applied to $\mathbb{C} :=$ Acc the ∞-category of accessible ∞-categories and accessible functors, $S :=$ N (Fin$_*$), $X := \mathcal{C}^{\otimes}$, and E being the class of inert morphisms in N (Fin$_*$), to get precisely the fact that CMon $(\mathcal{C}^{\otimes})$ is accessible. $\square$

6.12 Internal homs and duality

A very important part of any categorical model of Linear Logic is the internal hom representing the linear implication $A \multimap B$. In this section we review internal homs in the ∞-categorical setting and set up the theory of ∗-autonomous ∞-categories. As an important particular case, we discuss Lurie's theory of duals in a symmetric monoidal ∞-category and show that every symmetric monoidal ∞-category with duals is ∗-autonomous.

Definition 6.12.1. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_{*})$ be a symmetric monoidal ∞-category, and x, y be objects of $\mathcal{C}$. An *internal hom* from x to y in $\mathcal{C}$ is an object $[x, y]$ in $\mathcal{C}$ together with a morphism

$$\mathrm{ev} : x \otimes [x, y] \to y$$

such that for every $z \in \mathcal{C}_0$, the composite functor

$$\mathrm{Hom}_{\mathcal{C}}(z, [x, y]) \xrightarrow{x \otimes -} \mathrm{Hom}_{\mathcal{C}}(x \otimes z, x \otimes [x, y]) \xrightarrow{\mathrm{ev} \circ -} \mathrm{Hom}_{\mathcal{C}}(x \otimes z, y)$$

is a homotopy equivalence. In other words, $([x, y], \mathrm{ev})$ represents the presheaf $\mathrm{Hom}(x \otimes -, y) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$.

Fixing x, if there is an internal hom $[x, y]$ for every $y \in \mathcal{C}_0$, x is said to be *exponentiable*. If every object in $\mathcal{C}$ is exponentiable, $\mathcal{C}^{\otimes}$ is is said to be *closed symmetric monoidal*. If the monoidal structure is moreover cartesian, we say that $\mathcal{C}$ is *cartesian closed*.

Remark 6.12.2. Beware not to confuse this notion of exponentiability with the exponential of Linear Logic.

Note that by fact 6.7.12, x is exponentiable if and only if the functor $x \otimes - : \mathcal{C} \to \mathcal{C}$ admits a right adjoint $[x, -]$ (this is actually the definition of "monoidal closed" given by Lurie in [Lur17, Def 4.1.1.15]). In particular, in this situation the internal homs $[x, y]$ can be taken to be functorial in y using this right adjoint.

Remark 6.12.3. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category such that $\mathcal{C}$ is presentable. Then by fact 6.8.15, $\mathcal{C}^{\otimes}$ is closed if and only if for every $x \in \mathcal{C}_0$, the functor $x \otimes - : \mathcal{C} \to \mathcal{C}$ preserves small colimits, in which case $\mathcal{C}^{\otimes}$ is said to be *presentably symmetric monoidal*.

In particular, by fact 6.11.20, presentably symmetric monoidal ∞-categories admit free commutative monoids.

In the case that every $x \in \mathcal{C}_0$ is exponentiable, we can also get a functoriality in x:

Fact 6.12.4 ([Lur17, Rmk 4.6.1.4]). *In a closed symmetric monoidal ∞-category $\mathcal{C}$, the internal hom extends to a functor of two variables, denoted*

$$\underline{\mathrm{Hom}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}.$$

Proof. Following the discussion before the proposition, $\mathcal{C}^{\otimes}$ being closed implies that for every $y \in \mathcal{C}_0$, the functor

$$\mathrm{Hom}(x \otimes -, y) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$$

is representable. In particular, the functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$$
$$(x, y) \mapsto \mathrm{Hom}(x \otimes -, y)$$

lands in the essential image $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{o}p, \mathcal{S})$ of the Yoneda embedding $\unrhd_{\mathcal{C}}$. We can thus post-compose with an inverse to the equivalence $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of corollary 6.5.16 and get a functor

$$\underline{\mathrm{Hom}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for every $x \in \mathcal{C}_0$, $(x \otimes -) \dashv \underline{\mathrm{Hom}}(x, -)$. $\qquad\square$

We can use this functoriality to prove that the contravariant internal hom functor $\underline{\mathrm{Hom}}(-, y)$ is self-adjoint:

Proposition 6.12.5. *Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category, and y an object of $\mathcal{C}$. There is an adjunction*

$$\mathcal{C} \overset{\underline{\mathrm{Hom}}(-,y)^{\mathrm{op}}}{\underset{\underline{\mathrm{Hom}}(-,y)}{\rightleftarrows}} \mathcal{C}^{\mathrm{op}} \qquad \bot$$

Proof. There is a chain of homotopy equivalences

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(a, \underline{\mathrm{Hom}}(b, y)) &\simeq \mathrm{Hom}_{\mathcal{C}}(b \otimes a, y) & (b \otimes -) \dashv \underline{\mathrm{Hom}}(b, -) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(a \otimes b, y) & (b \otimes a \simeq a \otimes b) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(b, \underline{\mathrm{Hom}}(a, y)) & (a \otimes -) \dashv \underline{\mathrm{Hom}}(a, -) \\
&\simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\underline{\mathrm{Hom}}(a, y), b)
\end{aligned}
$$

natural in $a, b \in \mathcal{C}_0$. $\qquad\square$

This leads us to the definitions of dualizing objects and $*$-autonomy:

Definition 6.12.6. An object $\bot$ in a closed symmetric monoidal ∞-category is said to be *dualizing* if the contravariant internal hom functor

$$\underline{\mathrm{Hom}}(-, \bot) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

is an equivalence of ∞-categories.

A $*$-*autonomous* ∞-category is a pair $(\mathcal{C}^{\otimes}, \bot)$ with $\mathcal{C}^{\otimes}$ a closed symmetric monoidal ∞-category and $\bot \in \mathcal{C}_0$ a dualizing object. In a $*$-autonomous ∞-category $(\mathcal{C}^{\otimes}, \bot)$, we write $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ for the functor $\underline{\mathrm{Hom}}(-, \bot)$.

Proposition 6.12.7. *Let $\mathcal{C}^{\otimes}$ be a closed symmetric monoidal ∞-category and $\bot$ an object of $\mathcal{C}$. Then $\bot$ is dualizing if and only if for every object $x \in \mathcal{C}_0$, the morphism*

$$x \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(x, \bot), \bot)$$

given by the unit of the adjunction $\underline{\mathrm{Hom}}(-, \bot)^{\mathrm{op}} \dashv \underline{\mathrm{Hom}}(-, \bot)$ of proposition 6.12.5 is an isomorphism.

Proof. By proposition 6.7.11, $\underline{\mathrm{Hom}}(-, \bot)$ is an equivalence if and only if the unit and counit of the adjunction $\underline{\mathrm{Hom}}(-, \bot)^{\mathrm{op}} \dashv \underline{\mathrm{Hom}}(-, \bot)$ are both isomorphisms. But the counit of this adjunction is precisely its unit seen in the opposite ∞-category, so it is an isomorphism if and only if the unit is, which concludes the proof. $\qquad\square$

In some situations, the unit $\mathbf{1} \in \mathcal{C}_0$ of the symmetric monoidal ∞-category $\mathcal{C}$ can be a dualizing object. This occurs typically when $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category with *duals*.

Definition 6.12.8 ([Lur17, Def 4.6.1.1]). Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and x an object of $\mathcal{C}$. A *dual* of x in $\mathcal{C}^{\otimes}$ is an object $x^{\vee}$ together with morphisms

$$c : \mathbf{1} \to x \otimes x^{\vee} \quad e : x^{\vee} \otimes x \to I$$

such that the following diagrams in $\mathcal{C}$ commute up to homotopy:

An object that admits a dual is said to be *dualizable*. If every object in $\mathcal{C}$ is dualizable, we say that $\mathcal{C}^{\otimes}$ is *compact closed*.

Remark 6.12.9. In the literature, compact closed categories are also called *rigid* categories, although this appellation seems to be more frequent in the non-symmetric case.

Remark 6.12.10. Notice the similarity between definition 6.12.8 and item 4 of fact 6.7.1. This similarity can be made more formal through the notion of *adjunction in a bicategory*: unfolding definitions, adjunctions between ∞-categories are exactly adjunctions in the bicategory of quasicategories, functors and homotopy classes of natural transformations. On the other hand, every symmetric monoidal category $\mathcal{C}$ an be seen as a bicategory with one object, and the adjoint morphisms in this bicategory are precisely the dual objects in $\mathcal{C}$.

Fact 6.12.11 ([Lur17, Lem 4.6.1.5]). *Let $\mathcal{C}^{\otimes}$ be a closed symmetric monoidal ∞-category, x, y objects of $\mathcal{C}$ and $(x^{\vee}, c, e)$ a dual of x. Then the morphism*

$$(y \otimes x^{\vee}) \otimes x \simeq y \otimes (x^{\vee} \otimes x) \xrightarrow{e} y \otimes \mathbf{1} \simeq y$$

exhibits $y \otimes x^{\vee}$ as an internal hom from x to y in $\mathcal{C}$.

In particular, if $\mathcal{C}^{\otimes}$ is compact closed, then it is closed symmetric monoidal, with

$$\underline{\mathrm{Hom}}(x, y) := y \otimes x^{\vee}.$$

In particular, the object $x^{\vee}$ is unique up to a contractible space of choices when it exists, since it represents the presheaf

$$\mathrm{Hom}_{\mathcal{C}}(x \otimes -, \mathbf{1}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}.$$

The following fact is due to Yonatan Harpaz in a webpost on mathoverflow, and seems so far unpublished:

Fact 6.12.12 ([Har18]). *If $\mathcal{C}^{\otimes}$ is a compact closed ∞-category, then the assignment $x \mapsto x^{\vee}$ defines a contravariant self-equivalence*

$$(-)^{\vee} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}.$$

Corollary 6.12.13. *If $\mathcal{C}^{\otimes}$ is a compact closed ∞-category, then $\mathbf{1}$ is a dualizing object in $\mathcal{C}$ and $(\mathcal{C}^{\otimes}, \mathbf{1})$ is a $*$-autonomous ∞-category.*

6.13 Monads and their algebras

The final ingredient we need to talk about ∞-categorical models of Linear Logic is a theory of monads. Monads are usually defined as monoids in Fun $(\mathcal{C}, \mathcal{C})$, with tensor product given by the composition of functors. Since the composition of functors is clearly not commutative, we need a notion of non-symmetric monoidal structure and non-commutative monoids therein. As we mentioned in remark 6.11.7, this is achieved in Lurie's framework by replacing the ∞-operad $\mathrm{N}\,(\mathrm{Fin}_*)$ in the definition of symmetric monoidal structures by an arbitrary ∞-operad.

6.13.1 $\mathcal{O}$-monoidal ∞-categories and monads

Definition 6.13.1 ([Lur17, Def 2.1.2.13])**.** Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A $\mathcal{O}$-monoidal ∞-category is a cocartesian fibration $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ such that the composite functor

$$\mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$$

exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

Remark 6.13.2 ([Lur17, Prop 2.1.2.12])**.** A cocartesian fibration $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a $\mathcal{O}$-monoidal ∞-category if and only if for every $o = (o_1, \ldots, o_n)$ in $\mathcal{O}^{\otimes}_{\langle n \rangle}$ under the identification $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^n$, the inert morphisms $o \to o_i$ in $\mathcal{O}^{\otimes}$ induce an equivalence of ∞-categories $\mathcal{C}^{\otimes}_o \simeq \prod_{1 \leq i \leq n} \mathcal{C}^{\otimes}_{o_i}$.

Remark 6.13.3. Given an $\mathcal{O}$-monoidal ∞-category $\mathcal{C}^{\otimes}$, we can apply the same reasoning as in example 6.10.12 to get an $\mathcal{O}$-monoidal structure $(\mathcal{C}^{\mathrm{op}})^{\otimes}$ on $\mathcal{C}^{\mathrm{op}}$, and this induces an equivalence between $\mathcal{O}$-monoidal structures on $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$.

Definition 6.13.4. Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}$-monoidal ∞-category. An $\mathcal{O}$-algebra in $\mathcal{C}^{\otimes}$ is a functor $M : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ such that $p \circ M = \mathrm{id}_{\mathcal{C}^{\otimes}}$ and M maps inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{C}^{\otimes}$. The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$ of such functors.

Example 6.13.5. The $\mathrm{N}\,(\mathrm{Fin}_*)$-monoidal ∞-categories are precisely the symmetric monoidal ∞-categories in the sense of definition 6.10.4, and the $\mathrm{N}\,(\mathrm{Fin}_*)$-algebras are precisely the commutative monoids.

Definition 6.13.6 ([Lur17, Def 4.1.1.3])**.** Let $\mathrm{Assoc}^{\otimes}$ denote the nerve of the following category:

- The objects are the same as Fin_*.

- A morphism from $\langle m \rangle$ to $\langle n \rangle$ is a pair $(f, (\leq_i)_{1 \leq i \leq n})$ of a morphism $f : \langle m \rangle \to \langle n \rangle$ in Fin_* together with, for every $1 \leq i \leq n$, a linear ordering $\leq_i$ on $f^{-1}(i) \subseteq \langle m \rangle$.

- the linear ordering on the fibers of the composition of morphisms is given by lexicographic ordering.

Then the canonical functor $\mathrm{Assoc}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ exhibits $\mathrm{Assoc}^{\otimes}$ as an ∞-operad.

Definition 6.13.7. The Assoc-monoidal ∞-categories are simply called *monoidal* ∞-categories, and the Assoc-algebras are called *monoids*. We write $\mathrm{Mon}\,(\mathcal{C}^{\otimes}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\otimes})$.

Let $p : \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a monoidal ∞-category. There are two morphisms $\langle 2 \rangle \to \langle 1 \rangle$ in $\mathrm{Assoc}^{\otimes}$ whose underlying morphism in $\mathrm{N}\,(\mathrm{Fin}_*)$ is $1, 2 \mapsto 1$, corresponding to the two ways of linearly ordering the set $\{1, 2\}$. These two morphisms induce two functors

$$- \otimes_L - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
$$- \otimes_R - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

Composing with the transposition morphism $\langle 2 \rangle \to \langle 2 \rangle$, we can show that there are isomorphisms in $\mathcal{C} := \mathcal{C}_{\langle 1 \rangle}^{\otimes}$

$$X \otimes_L Y \simeq Y \otimes_R X$$

natural in X and Y, but in general there doesn't have to be any isomorphism $X \otimes_L Y \simeq Y \otimes_R X$, as one would expect in a non-symmetric monoidal structure.

Definition 6.13.8. Given a monoidal ∞-category $\mathcal{C}^{\otimes}$, we write $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for the functor induced by the morphism $f : \langle 2 \rangle \to \langle 1 \rangle$ in Assoc$^{\otimes}$ given by $1, 2 \mapsto 1$, and whose linear ordering on $f^{-1}(1) = \{1, 2\}$ is given by $1 < 2$. This functor is called the *tensor product* of $\mathcal{C}^{\otimes}$.

The unique map $\langle 0 \rangle \to \langle 1 \rangle$ in Assoc$^{\otimes}$ induces an object $\mathbf{1} \in \mathcal{C}_0$ called the unit object of $\mathcal{C}^{\otimes}$.

Example 6.13.9. Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category. Then the functor $\pi_2 : \mathcal{C}^{\otimes} \times_{\mathrm{N}\,(\mathrm{Fin}_*)} \mathrm{Assoc}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ defines a monoidal structure on $\mathcal{C}$: this is the *underlying monoidal structure* of the symmetric monoidal structure p.

Most of the results of sections 6.10 to 6.12 can be adapted to work for monoidal ∞-categories, with some subtleties here and there (for instance, the formula for free monoids will not involve quotients by $\mathfrak{S}_n$, there may in general be two internal homs depending on whether they are right adjoint to $X \otimes -$ or $- \otimes X$, etc.). We will not repeat such results here, instead focusing on the application of monoidal structures to the ∞-categorical theory of monads.

Definition 6.13.10. The 1-category qCat is cartesian closed, so for every ∞-category $\mathcal{C}$, the functor ∞-category Fun $(\mathcal{C}, \mathcal{C})$ can be endowed with the structure of a monoid in the cartesian 1-category qCat with tensor product given by the composition of functors, hence a monoid in the monoidal ∞-category qCat$^{\times}$. Composing with the canonical functor $\mathrm{N}\,(\mathrm{qCat}) \to \infty\mathrm{Cat}$, we get a monoid in $\infty\mathrm{Cat}$, which by fact 6.4.27 is equivalent to a monoidal structure on Fun $(\mathcal{C}, \mathcal{C})$, which we write Fun $(\mathcal{C}, \mathcal{C})^{\circ}$.

The monoids in the monoidal ∞-category Fun $(\mathcal{C}, \mathcal{C})^{\circ}$ are called *monads* on $\mathcal{C}$, and we write

$$\mathrm{Mnd}\,(\mathcal{C}) := \mathrm{Mon}\,(\mathrm{Fun}\,(\mathcal{C}, \mathcal{C})^{\circ}).$$

Remark 6.13.11. What makes this definition work is the fact that the composition of functors in $\infty\mathrm{Cat}$ is "strictly associative" rather than simply "homotopy associative" when working with ∞-categories defined as simplicial sets.

6.13.2 Algebras over monads

We now need to define *algebras over a monad*. Let $T : \mathcal{C} \to \mathcal{C}$ be (the underlying functor of) a monad on $\mathcal{C}$. A T-algebra in $\mathcal{C}$ should in particular induce an object $X \in \mathcal{C}_0$ together with a morphism $T(X) \to X$, such that the two induced morphism $T(T(X)) \to X$ coincide up to homotopy.

To formalize this, Lurie relies on the more general notion of *left module*: given a monoidal category $(\mathcal{C}, \otimes)$, one can define a *left action* of $\mathcal{C}$ on a category $\mathcal{D}$ to be a functor

$$- \star - : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

together with natural isomorphisms witnessing the usual axioms of monoid left actions. Then, given a monoid M in $\mathcal{C}$, one can define a *left M-module* in $\mathcal{D}$ to be an object of X in $\mathcal{D}$ together

with a morphism $f : M \star X \to X$ such that the following diagram commutes:

$$(M \otimes M) \star X \xrightarrow{\simeq} M \star (M \star X) \xrightarrow{\mathrm{id}_M \star f} M \star X \tag{6.6}$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow f$$
$$M \star X \xrightarrow{\hspace{6cm} f} X$$

Remark 6.13.12. Such actions of monoidal categories have also been called *actegories* in the literature [CG24].

The notion of T-algebras for a monad $\mathcal{C} \to \mathcal{C}$ can then be recovered as the left T-modules in $\mathcal{C}$ for the algebra T in $(\mathrm{Fun}\,(\mathcal{C},\mathcal{C}), \circ)$, where the action of categories is given by

$$\mathrm{Fun}\,(\mathcal{C},\mathcal{C}) \times \mathcal{C} \to \mathcal{C}$$
$$(F, X) \mapsto F(X)$$

In the ∞-categorical, such a structure of monoidal category action is represented using an ∞-operad $\mathrm{LM}^{\otimes}$. We do not give the precise definition of LM here, referring the interested reader to [Lur17, Section 4.2.1], and only give some intuitions of what it represents:

Fact 6.13.13. *The underlying ∞-category* $\mathrm{LM} := \mathrm{LM}^{\otimes}_{\langle 1 \rangle}$ *of the ∞-operad* $\mathrm{LM}^{\otimes}$ *is discrete on on two objects, denoted* $\mathfrak{a}$ *and* $\mathfrak{m}$. $\mathrm{LM}^{\otimes}$ *contains* $\mathrm{Assoc}^{\otimes} \subseteq \mathrm{LM}^{\otimes}$ *as the full subcategory spanned by the tuples in* $\mathrm{LM}^n \simeq \mathrm{LM}^{\otimes}_{\langle 1 \rangle}$ *only containing the object* $\mathfrak{a}$.

If $p : \mathcal{C}^{\otimes} \to \mathrm{LM}^{\otimes}$ *is an LM-monoidal ∞-category, then the inclusion* $\mathrm{Assoc}^{\otimes} \subseteq \mathrm{LM}^{\otimes}$ *induces a monoidal structure on* $\mathcal{C}^{\otimes}_{\mathfrak{a}}$, *and* $\mathcal{C}^{\otimes}$ *can be thought of as the data of this monoidal ∞-category* $\mathcal{C}^{\otimes}_{\mathfrak{a}}$ *together with an action on the ∞-category* $\mathcal{C}^{\otimes}_{\mathfrak{m}}$. *In other words,* $\mathcal{C}^{\otimes}$ *comes equipped with a functor*

$$- \star - : \mathcal{C}^{\otimes}_{\mathfrak{a}} \times \mathcal{C}^{\otimes}_{\mathfrak{m}} \to \mathcal{C}^{\otimes}_{\mathfrak{m}}$$

that behaves as a homotopy-coherent left action.

An LM-algebra induces a monoid M *in* $\mathcal{C}^{\otimes}_{\mathfrak{a}}$, *an object* X *in* $\mathcal{C}^{\otimes}_{\mathfrak{m}}$ *and a morphism* $f : M \star X \to X$ *in* $\mathcal{C}^{\otimes}_{\mathfrak{m}}$ *such that the above diagram 6.6 commutes up to homotopy, together with higher-dimensional coherences.*

Definition 6.13.14 ([Lur17, Def 4.2.1.12]). Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category and $\mathcal{D}$ an ∞-category. A *left action* of $\mathcal{C}^{\otimes}$ on $\mathcal{D}$ is an LM-monoidal ∞-category $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes}$ together with equivalences

$$\mathcal{C}^{\otimes} \simeq \mathcal{M}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathrm{Assoc}^{\otimes}$$
$$\mathcal{D} \simeq \mathcal{M}^{\otimes}_{\mathfrak{m}}$$

(where the first equivalence is compatible with the monoidal structure).

Definition 6.13.15 ([Lur17, Def 4.2.1.13]). Given a left action of $\mathcal{C}^{\otimes}$ on $\mathcal{D}$, we write $\mathrm{LMod}\,(\mathcal{D}) := \mathrm{Alg}_{\mathrm{LM}}(\mathcal{M})$ for the ∞-category of *left module objects of* $\mathcal{M}$. The inclusion $\mathrm{Assoc}^{\otimes} \subseteq \mathrm{LM}^{\otimes}$ induces by precomposition a forgetful isofibration

$$\mathrm{LMod}\,(\mathcal{D}) \to \mathrm{Mon}\,(\mathcal{C}^{\otimes}).$$

Given a monoid M in $\mathcal{C}^{\otimes}$, we write

$$\mathrm{LMod}_M(\mathcal{D}) := \mathrm{LMod}\,(\mathcal{D}) \times_{\mathrm{Mon}\,(\mathcal{C})} \{M\}.$$

The ∞-category LMod_M is called the ∞-category of *left M-modules* in $\mathcal{D}$.

Definition 6.13.16. Definition 6.13.10 can be extended so that every ∞-category $\mathcal{C}$ induces an LM -monoidal ∞-category $\mathcal{M}^{\otimes}$ whose restriction to Assoc$^{\otimes}$ is Fun $(\mathcal{C},\mathcal{C})^{\circ}$ and that $\mathcal{M}_{\mathfrak{m}}^{\otimes} \simeq \mathcal{C}$, with the action

$$\text{Fun}\,(\mathcal{C},\mathcal{C}) \times \mathcal{C} \to \mathcal{C}$$

given by $(F, X) \mapsto F(X)$.

Given a monad T on $\mathcal{C}$, the objects of LMod $_T(\mathcal{C})$ are called the *T-algebras* in $\mathcal{C}$, and we write

$$\mathcal{C}^T := \text{LMod}\,_T(\mathcal{C}).$$

Fact 6.13.17 ([Lur17, Prop 4.2.4.2])**.** *Let $\mathcal{C}^{\otimes}$ be a monoidal ∞-category with a left action on an ∞-category $\mathcal{D}$. Then for every monoid M in $\mathcal{C}^{\otimes}$ and object X in $\mathcal{D}$, the object $|M| \star X$ in $\mathcal{D}$ admits a canonical left M-module structure, called the* free M-module *on X. The assignment $X \mapsto M \star X$ defines a functor $\mathcal{D} \to \text{LMod}\,_M(\mathcal{D})$ which is left adjoint to the forgetful functor $\text{LMod}\,_M(\mathcal{D}) \to \mathcal{D}$.*

Definition 6.13.18. Let T be a monad on an ∞-category $\mathcal{C}$, and X be an object of $\mathcal{C}$. Then the T-algebra $T(X)$ is called the *free T-algebra* on X. The essential image of the functor $T(-) : \mathcal{C} \to \mathcal{C}^T$ is called the *Kleisli ∞-category* of the monad T, and it is denoted $\mathcal{C}_T$.

6.13.3   Monads and adjunctions

As we have just seen, any monad T on an ∞-category $\mathcal{C}$ induces an adjunction

$$\mathcal{C} \xrightarrow[\substack{\longleftarrow \\ |-|}]{\substack{T \\ \perp}} \mathcal{C}^T$$

In the 1-categorical setting, any adjunction

$$\mathcal{C} \xrightarrow[\substack{\longleftarrow \\ R}]{\substack{L \\ \perp}} \mathcal{D}$$

between categories induces a monad $R \circ L$ on $\mathcal{C}$, with multiplication given by

$$RLRL \xrightarrow{R\varepsilon L} RL$$

where $\varepsilon : LR \to \text{id}_{\mathcal{D}}$ is the counit of the adjunction.

The same is also true in the ∞-categorical setting, although the monad structure is more complicated to construct due to all the higher coherences required. In this section we will only list facts about the relation between monads and adjunctions, without explicitly detailing the constructions involved, as we will not use such constructions directly, referring the interested reader to [Lur17, Section 4.7] and [MH25].

Fact 6.13.19 ([Lur17, Prop 4.7.3.3])**.** *Let*

$$\mathcal{C} \xrightarrow[\substack{\longleftarrow \\ R}]{\substack{L \\ \perp}} \mathcal{D}$$

be an adjunction between ∞-categories. Then there is monad structure on $R \circ L$ given informally as above, and a canonical functor

$$R' : \mathcal{D} \to \mathcal{C}^{R \circ L}$$

such that $|-| \circ R' : \mathcal{D} \to \mathcal{C}$ is naturally isomorphic to R.

Definition 6.13.20 ([Lur17, Def 4.7.3.4]). If the functor $R' : \mathcal{D} \to \mathcal{C}^{R \circ L}$ of fact 6.13.19 is an equivalence, we say that R is a *monadic right adjoint*, and that $\mathcal{D}$ is *monadic over* $\mathcal{C}$.

The following fact says that the theory of monads can entirely be understood in terms of the adjunctions they induce:

Fact 6.13.21 ([MH25, Thm 3.22]). *Let $\mathcal{C}$ be an ∞-category. The mapping $T \mapsto \mathcal{C}^T$ extends to a functor*

$$\mathrm{Mnd}\,(\mathcal{C})^{\mathrm{op}} \to \infty\,\mathrm{Cat}\,_{/\mathcal{C}}$$
$$T \mapsto (|-| : \mathcal{C}^T \to \mathcal{C})$$

This functor is fully faithful and its essential image consists of the monadic right adjoint functors.

A similar but dual statement also holds about Kleisli ∞-categories:

Fact 6.13.22 ([MH25, Thm 7.2]). *Let $\mathcal{C}$ be an ∞-category. The mapping $T \mapsto \mathcal{C}_T$ extends to a functor*

$$\mathrm{Mnd}\,(\mathcal{C}) \to \infty\,\mathrm{Cat}\,_{\mathcal{C}/}$$
$$T \mapsto (T(-) : \mathcal{C} \to \mathcal{C}_T)$$

(where $\infty\,\mathrm{Cat}\,_{\mathcal{C}/}$ denotes the coslice ∞-category of $\infty\,\mathrm{Cat}$ under $\mathcal{C}$). This functor is fully faithful, and its essential image consists of the essentially surjective left adjoint functors.

In particular, if $L : \mathcal{C} \to \mathcal{D}$ is any essentially surjective functor that admits a right adjoint R, then there is a canonical equivalence of ∞-categories

$$\mathcal{C}_T \simeq \mathcal{D}.$$

Chapter 7

Linear logic in ∞-categories

We are finally ready to tackle the ∞-categorical variants to the various models of chapter 3 we promised in the introduction! We begin in section 7.1 by definition linear/non-linear adjunctions between ∞-categories, and we dualize the theory commutative monoids in symmetric monoidal ∞-categories to obtain a well-behaved theory of commutative comonoids, using cofree commutative comonoids to define Lafont ∞-categories. In section 7.2, we leverage the extensional point of view on profunctors to define the ∞-category $\infty\operatorname{Prof}$ of profunctors as having presheaf ∞-categories as objects and cocontinuous functors as morphisms. We review Lurie's tensor product of ∞-categories with colimits, which we use to describe a closed symmetric monoidal structure on $\infty\operatorname{Prof}$, on which we build various linear/non-linear adjunctions by analogy with the generalized species and the Scott exponential of section 3.11. Finally, we introduce the theory of spectra, ring spectra and module spectra in section 7.3 as generalizations of commutative groups, rings and modules, and define construct ∞-categorical counterparts to the linear algebraic models of linear logic we reviewed in section 3.10.

7.1   Axiomatizations for ∞-categorical models

Throughout sections 6.9 to 6.13, we reviewed Lurie's formalism for higher symmetric and non-symmetric monoidal structures, as well as the theory of (commutative) monoids, monads and their algebras. But in Linear Logic, we are more interested in comonoids, comonads and their coalgebras: rather than rebuild the entire theory from this opposite perspective, we can rely on example 6.10.12 and remark 6.13.3 to get the right notions for free:

Definition 7.1.1. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and $\mathcal{C}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. An $\mathcal{O}$-*coalgebra* in $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-algebra in $(\mathcal{C}^{\mathrm{op}})^{\otimes}$. The ∞-category of $\mathcal{O}$-coalgebras in $\mathcal{C}^{\otimes}$ is defined as

$$\operatorname{Coalg}_{\mathcal{O}}(\mathcal{C}^{\otimes}) := (\operatorname{Alg}_{\mathcal{O}}((\mathcal{C}^{\mathrm{op}})^{\otimes}))^{\mathrm{op}}$$

In particular:

- when $\mathcal{O}^{\otimes} = \mathrm{N}\,(\mathrm{Fin}_*)$, $\mathcal{O}$-coalgebras are called *commutative comonoids* and we write $\operatorname{Comon}(\mathcal{C}^{\otimes}) := \operatorname{Coalg}_{\mathrm{N}\,(\mathrm{Fin}_*)}(\mathcal{C}^{\otimes})$,

- when $\mathcal{O}^{\otimes} = \operatorname{Assoc}^{\otimes}$ and $\mathcal{C}^{\otimes} = \operatorname{Fun}(\mathcal{D}, \mathcal{D})^{\circ}$, we speak of *comonads* on $\mathcal{D}$,

- when $\mathcal{O}^{\otimes} = \mathrm{LM}^{\otimes}$ and $\mathcal{C}^{\otimes}$ is the left action of $\mathrm{Fun}\,(\mathcal{D},\mathcal{D})^{\circ}$ on $\mathcal{D}$ from definition 6.13.16 and T is a comonad on $\mathcal{D}$, we write

$$\mathcal{D}^{T} := \mathrm{Coalg}_{\mathrm{LM}}(\mathcal{C}^{\otimes}) \times_{\mathrm{Coalg}_{\mathrm{Assoc}}(\mathrm{Fun}\,(\mathcal{D},\mathcal{D})^{\circ})} \{T\}$$

for the ∞-category of *T-coalgebras*.

Remark 7.1.2. Because of the isomorphism of simplicial sets

$$\mathrm{Fun}\,(\mathcal{C},\mathcal{C})^{\mathrm{op}} \simeq \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}},\mathcal{C}^{\mathrm{op}})^{\mathrm{op}},$$

comonads on $\mathcal{C}$ correspond precisely to monads on $\mathcal{C}^{\mathrm{op}}$. If $T : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a monad on $\mathcal{C}^{\mathrm{op}}$, writing $T^{\mathrm{op}} : \mathcal{C} \to \mathcal{C}$ for the corresponding comonad on $\mathcal{C}$, there is an isomorphism of simplicial sets (so in particular an equivalence of ∞-categories):

$$\mathcal{C}^{(T^{\mathrm{op}})} \simeq ((\mathcal{C}^{\mathrm{op}})^{T})^{\mathrm{op}}.$$

In other words, the ∞-category of T^{op}-coalgebras on $\mathcal{C}$ is equivalent to the opposite of the ∞-category of T-algebras on $\mathcal{C}^{\mathrm{op}}$.

Using remark 6.7.5, we can dualize the results of section 6.13.3 to formulate a dual correspondence between comonads and adjunctions:

Definition 7.1.3. If T is a comonad on an ∞-category $\mathcal{C}$, then for every object $X \in \mathcal{C}_0$, the object $T(X)$ inherits a canonical T-coalgebra structure, called the *cofree T-algebra* on X. This induces a right adjoint $T(-)$ to the forgetful functor $\mathcal{C}^{T} \to \mathcal{C}$, whose essential image is called the *Kleisli ∞-category of the comonad T*, and denoted $\mathcal{C}_{T}$.

Proposition 7.1.4. *Let*

$$\mathcal{C} \; \overset{L}{\underset{R}{\xrightarrow{\;\perp\;}}} \; \mathcal{D}$$

be an adjunction between ∞-categories. Then there is comonad structure on $L \circ R : \mathcal{D} \to \mathcal{D}$, and a canonical functor

$$L' : \mathcal{C} \to \mathcal{C}^{L \circ R}$$

such that $|-| \circ L' : \mathcal{C} \to \mathcal{D}$ is naturally isomorphic to L.

Moreover, the essential image of R in $\mathcal{C}$ is canonically equivalent to the Kleisli ∞-category $\mathcal{D}_{Ł \circ R}$.

Definition 7.1.5. If the functor $L' : \mathcal{D} \to \mathcal{C}^{R \circ L}$ of proposition 7.1.4 is an equivalence, we say that L is a *comonadic left adjoint*, and that $\mathcal{C}$ is *comonadic over $\mathcal{D}$*.

Proposition 7.1.6. *Given an ∞-category $\mathcal{D}$, the following data are equivalent:*

1. *the data of a comonad on $\mathcal{D}$,*

2. *the data of a comonadic left adjoint $\mathcal{C} \to \mathcal{D}$,*

3. *the data of an essentially surjective right adjoint $\mathcal{D} \to \mathcal{C}'$.*

We are now well equipped to define ∞-categorical to the axiomatizations of categorical models of Linear Logic we review in section 3.4:

Definition 7.1.7. An ∞-categorical *linear/non-linear adjunction* is the data of:

1. a closed symmetric monoidal ∞-category $\mathcal{L}^{\otimes}$

2. a cartesian symmetric monoidal ∞-category $\mathcal{M}^\times$

3. a (strongly) symmetric monoidal functor $L : \mathcal{M}^\times \to \mathcal{L}^\otimes$

4. such that the underlying functor $L : \mathcal{M} \to \mathcal{L}$ admits a right adjoint $M : \mathcal{L} \to \mathcal{M}$.

$$\mathcal{M} \underset{M}{\overset{L}{\rightleftarrows}} \mathcal{L}$$

By fact 6.11.13, the right adjoint $M : \mathcal{L} \to \mathcal{M}$ in definition 7.1.7 is necessarily lax symmetric monoidal, so in particular the underlying functor of the induced comonad $L \circ M : \mathcal{L} \to \mathcal{L}$ is lax monoidal, as one would expect from the definition of linear categories.

Remark 7.1.8. Just like in the 1-categorical setting and the homotopy type theoretical setting, we can moreover say that a linear/non-linear adjunction models the additive connectives when $\mathcal{L}$ admits finite products and finite coproducts, and that it is a model of classical linear logic if it is moreover ∗-autonomous.

However, we are unsure of how to properly adapt definition 3.4.7 to the ∞-categorical setting. We could define a *lax symmetric monoidal comonad* ! to be a comonoid in the ∞-category $\text{Fun}^{\text{lax}}(\mathcal{C}^\otimes, \mathcal{C}^\otimes)^\circ$, but the additional data of a natural commutative comonoid structure on every $!X$ for $X \in \mathcal{C}_0$ whose structural morphisms are compatible with the !-coalgebra structure is a bit more elusive: it is unclear how to package such data in a homotopy-coherent manner. However, relying on the correspondence between comonads and comonadic left adjoints and on fact 3.4.11 that linear/non-linear adjunctions induce linear categories to have a candidate definition:

Open problem 7.1.9. Properly define *linear ∞-categories*, and show that the data of a linear ∞-category is equivalent to the data of a linear/non-linear adjunction whose left adjoint is comonadic.

The situation for Seely categories is even murkier because of axiom 5 in definition 3.4.3, requiring a specific diagram to be commutative will most certainly give the right homotopy-coherent definition. Again, we can use the correspondence between comonads and essentially surjective right adjoints:

Open problem 7.1.10. Define *Seely ∞-categories* and prove that the data of a Seely ∞-category is equivalent to the data of a linear/non-linear adjunction whose linear category $\mathcal{L}$ admits finite products and whose right adjoint is essentially surjective.

Instead, the only other definition that we can readily adapt is that of Lafont categories:

Definition 7.1.11. A *Lafont ∞-category* is a closed symmetric monoidal ∞-category $\mathcal{C}^\otimes$ such that the forgetful functor $\text{Comon}(\mathcal{C}^\otimes) \to \mathcal{C}$ admits a right adjoint.

Note that the existence of such a right adjoint is equivalent to the existence of a cofree commutative comonoid for every object, by the dual of remark 6.11.17.

Proposition 7.1.12. *Any Lafont ∞-category $\mathcal{C}^\otimes$ induces a linear/non-linear adjunction between $\mathcal{C}$ and* $\text{Comon}(\mathcal{C}^\otimes)$.

Proof. The dual statement of fact 6.11.5 precisely states that $\text{Comon}(\mathcal{C}^\otimes)$ is cartesian, and that the forgetful functor is symmetric monoidal from the cartesian structure on $\text{Comon}(\mathcal{C}^\otimes)$ to the symmetric monoidal structure on $\mathcal{C}$. That this functor moreover admits a right adjoint is precisely the remaining ingredient in the definition of linear/non-linear adjunction. □

In ordinary category theory, a sufficient condition for the existence of cofree commutative comonoids is that there is a sufficient supply of limits and that those limits are compatible with the tensor product. In the context of Linear Logic, this has been explored by Melliès, Tabareau and Tasson in [MTT09]. In the ∞-categorical context, we can get a similar result as a dual of facts 6.11.19 and 6.11.20:

Proposition 7.1.13. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category with countable limits such that, for every object X in $\mathcal{C}$, the tensor product functor $X \otimes - : \mathcal{C} \to \mathcal{C}$ preserves countable limits.*

Then $\mathcal{C}$ admits cofree commutative comonoids, and the cofree commutative comonoid on an object A is given by

$$\prod_{n \in \mathbb{N}} (A^{\otimes n})^{\mathfrak{S}_n},$$

where $(A^{\otimes n})^{\mathfrak{S}_n}$ denotes the limit of the diagram $\mathrm{B}\,\mathfrak{S}_n \to \mathcal{C}$ of definition 6.11.18.

Remark 7.1.14. In homotopy theory, the object $(A^{\otimes n})^{\mathfrak{S}_n}$ is classically called the object of *homotopy invariants* of the action of $\mathfrak{S}_n$ on $A^{\otimes n}$.

However this result is seldom applicable in practice: by fact 6.7.18, tensor products in closed symmetric monoidal ∞-categories are compatible with colimits, and not often with limits. As we now show, it is still possible to recover cofree commutative comonoids when the tensor product is only compatible with colimits, as long as the ∞-category considered is *presentable*:

Theorem 7.1.15. *If $\mathcal{C}^{\otimes}$ is a presentably symmetric monoidal ∞-category (remark 6.12.3), then $\mathrm{Comon}\,(\mathcal{C}^{\otimes})$ is presentable and $\mathcal{C}^{\otimes}$ admits cofree commutative comonoids, so in particular $\mathcal{C}^{\otimes}$ is a Lafont ∞-category.*

Note that $\mathrm{Comon}\,(\mathcal{C}^{\otimes})$ being presentable does not immediately follow from fact 6.11.26 by duality, since $\mathrm{Comon}\,(\mathcal{C}^{\otimes}) \simeq \mathrm{CMon}\,((\mathcal{C}^{\mathrm{op}})^{\otimes})^{\mathrm{op}}$, and the opposite of a presentable ∞-category is generally not itself presentable!

Proof of theorem 7.1.15. First suppose we have shown that $\mathrm{Comon}\,(\mathcal{C}^{\otimes})$ is accessible. By the dual statement of fact 6.11.24, $\mathrm{Comon}\,(\mathcal{C}^{\otimes})$ admits small colimits, so it is presentable, and the forgetful functor $\mathrm{Comon}\,(\mathcal{C}^{\otimes}) \to \mathcal{C}$ preserves small colimits, so by fact 6.8.15 it admits a right adjoint and $\mathcal{C}^{\otimes}$ is a Lafont ∞-category.

To show that $\mathrm{Comon}\,(\mathcal{C})$ is accessible, we will adapt the proof of fact 6.11.26. Say that an ∞-category $\mathcal{D}$ is *coaccessible* if $\mathcal{D}^{\mathrm{op}}$ is accessible, and similarly for *coaccessible functors*. Writing Coacc for the subcategory of $\infty\,\mathrm{Cat}$ spanned by the coaccessible ∞-categories and the coaccessible functors, we want to show that $\mathrm{CMon}\,((\mathcal{C}^{\mathrm{op}})^{\otimes})^{\mathrm{op}}$ belongs to Coacc. In order to do that, we apply [Lur09, Thm 5.4.7.11] to the cocartesian fibration $p : (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathrm{N}\,(\mathrm{Fin}_*)$ and the class of inert morphisms in $\mathrm{N}\,(\mathrm{Fin}_*)$, and are left with showing that Coacc and p satisfy the hypotheses of that theorem. We will use repeatedly the following fact:

(*) the self-equivalence $\mathrm{op} : \infty\,\mathrm{Cat} \simeq \infty\,\mathrm{Cat}$ restricts to an equivalence $\mathrm{op} : \mathrm{Coacc} \simeq \mathrm{Acc}$ (by definition of Coacc),

We first show that Coacc satisfies the hypotheses of [Lur09, Thm 5.4.7.11]:

(a) By [Lur09, Prop 5.4.7.3], Acc admits small limits and the forgetful functor $\mathrm{Acc} \to \mathrm{Cat}_{\mathcal{U}_1}$ preserves small limits. By commutativity of the diagram

$$
\begin{array}{ccc}
\mathrm{Coacc} & \longrightarrow & \infty\,\mathrm{Cat} \\
{\scriptstyle \mathrm{op}}\downarrow & & \downarrow{\scriptstyle \mathrm{op}} \\
\mathrm{Acc} & \longrightarrow & \infty\,\mathrm{Cat}
\end{array}
$$

it follows that Coacc also admits small limits and that the functor Coacc $\to \infty$ Cat preserves them.

(b) Given a coaccessible ∞-category $\mathcal{A}$, we have

$$\text{Fun}\,(\Delta^1, \mathcal{A})^{\text{op}} \simeq \text{Fun}\,(\Delta^1, \mathcal{A}^{\text{op}})$$

(because $(\Delta^1)^{\text{op}} \simeq \Delta^1$). Fun $(\Delta^1, \mathcal{A}^{\text{op}})$ is accessible by [Lur09, Prop 5.4.4.3], so Fun $(\Delta^1, \mathcal{A})$ is coaccessible.

(c) Given coaccessible ∞-categories $\mathcal{A}$ and $\mathcal{B}$, a functor $F : \mathcal{A} \to \text{Fun}\,(\Delta^1, \mathcal{B})$ is coaccessible if and only if the two functors $\mathcal{A} \to \mathcal{B}$ induced by restricting to the vertices of Δ^1 are coaccessible. This is because coaccessibility is a limit-preservation property, and limits in functor ∞-categories are computed pointwise (fact 6.6.30).

We now show that the functor $p : (\mathcal{C}^{\text{op}})^{\otimes} \to \text{N}\,(\text{Fin}_*)$ satisfies the hypotheses of the theorem.

(i) By definition of being a symmetric monoidal ∞-category, the functor $p : (\mathcal{C}^{\text{op}})^{\otimes} \to \text{N}\,(\text{Fin}_*)$ is a cocartesian fibration.

(ii) The fiber of p over $\langle n \rangle \in \text{N}\,(\text{Fin}_*)$ is isomorphic to $(\mathcal{C}^{\text{op}})^n$, which is coaccessible because $\mathcal{C}^{\text{op}}$ is coaccessible and by item (a).

(iii) Given a morphism $f : \langle m \rangle \to \langle n \rangle$ in $\text{N}\,(\text{Fin}_*)$, the induced covariant transport functor $f_! : (\mathcal{C}^{\text{op}})^m \to (\mathcal{C}^{\text{op}})^n$ is opposite to the functor $\mathcal{C}^m \to \mathcal{C}^n$ induced by the monoidal structure on $\mathcal{C}$. This latter functor is accessible by virtue of [Lur17, Lem 3.2.3.4], so $f_!$ is coaccessible.

By [Lur09, Thm 5.4.7.11], Mon $(\mathcal{C}^{\text{op}})$ is thus coaccessible, so that Comon $(\mathcal{C})$ is accessible, which concludes the proof. $\qquad\square$

Remark 7.1.16. Taking $\mathcal{C}$ to be the nerve of the category of R-modules for a ring R, theorem 7.1.15 recovers the Lafont exponential of proposition 3.10.8.

We conclude this section by noting that in the case of $*$-autonomous ∞-categories, the existence of free commutative monoids implies the existence of cofree commutative comonoids.

Proposition 7.1.17. *Let $(\mathcal{C}^{\otimes}, \perp)$ be a $*$-autonomous ∞-category. If the forgetful functor CMon $(\mathcal{C}^{\otimes}) \to \mathcal{C}$ admits a left adjoint L (i.e. $\mathcal{C}^{\otimes}$ admits free commutative monoids), then it is a Lafont ∞-category. Moreover, the underlying object of the cofree commutative comonoid on $X \in \mathcal{C}_0$ is given by $(L(X^*))^*$.*

Proof. Since $L : \mathcal{C} \to$ Mon $(\mathcal{C})$ is a left adjoint, $L^{\text{op}} : \mathcal{C}^{\text{op}} \to$ Mon $(\mathcal{C})^{\text{op}}$ is a right adjoint (remark 6.7.5). Using the equivalence $(-)^* : \mathcal{C}^{\text{op}} \simeq \mathcal{C}$ and passing to ∞-categories of commutative monoids, we get an equivalence

$$\text{Mon}\,(\mathcal{C})^{\text{op}} \simeq \text{Mon}\,(\mathcal{C}^{\text{op}})^{\text{op}} = \text{Comon}\,(\mathcal{C}).$$

Hence the composite functor

$$\text{Comon}\,(\mathcal{C}^{\otimes}) \xleftarrow{\quad = \quad} \text{CMon}\,((\mathcal{C}^{\text{op}})^{\otimes})^{\text{op}} \xleftarrow{\text{CMon}\,((-)^*)^{\text{op}}} \text{CMon}\,(\mathcal{C}^{\otimes})^{\text{op}} \underset{L^{\text{op}}}{\overset{\perp}{\rightleftarrows}} \mathcal{C}^{\text{op}} \xleftarrow{(-)^*} \mathcal{C}$$

defines a right adjoint to the forgetful functor Comon $(\mathcal{C}^{\otimes}) \to \mathcal{C}$. Unfolding this composition, we also get the desired formula for the cofree commutative comonoid on X. $\qquad\square$

7.2 Higher species and domains

In this section, we will give ∞-categorical generalizations to the 1-categorical models of relations and domains, and to their bicategorical variants we review in section 3.11. To that effect, we first review some very general results about ∞-categories with colimits, which will also prove to be very useful in the higher algebraic models of section 7.3.

7.2.1 ∞-categories with specified colimits

Definition 7.2.1. Let $\mathcal{K}$ be a class of small simplicial sets. Say that an ∞-category $\mathcal{C}$ is $\mathcal{K}$-*cocomplete* if it admits K-indexed colimits for every $K \in \mathcal{K}$, and that a functor $F : \mathcal{C} \to \mathcal{D}$ is $\mathcal{K}$-*cocontinuous* if it preserves K-indexed colimits for every $K \in \mathcal{K}$. We write

$$\infty\mathrm{Cat}(\mathcal{K}) \subseteq \infty\mathrm{Cat}_{\mathcal{U}_1}$$

for the subcategory spanned by the $\mathcal{K}$-cocomplete ∞-categories and the $\mathcal{K}$-cocontinuous functors (where $\infty\mathrm{Cat}_{\mathcal{U}_1}$ denotes the very large ∞-category of large ∞-categories). For every $\mathcal{C}, \mathcal{D}$ in $\infty\mathrm{Cat}(\mathcal{K})$, we write

$$\mathrm{Fun}_{\mathcal{K}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors that preserve K-indexed colimits

Definition 7.2.2. A functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ between $\mathcal{K}$-cocomplete ∞-categories is said to be $\mathcal{K}$-*bicontinuous* if for every objects $c \in \mathcal{C}_0, d \in \mathcal{D}_0$, the functors

$$F(c, -) : \mathcal{D} \to \mathcal{E}$$
$$F(-, d) : \mathcal{C} \to \mathcal{E}$$

are $\mathcal{K}$-cocontinuous. We write $\mathrm{Bicc}_{\mathcal{K}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for the full subcategory spanned by the $\mathcal{K}$-bicontinuous functors.

Here are the main results that we will use:

Fact 7.2.3 ([Lur17, Section 4.8.1]). *For every class $\mathcal{K}$ of small simplicial sets, the ∞-category $\infty\mathrm{Cat}(\mathcal{K})$ admits a closed symmetric monoidal structure whose tensor product classifies $\mathcal{K}$-bicontinuous functors, in the following sense: there is a $\mathcal{K}$-bicontinuous functor $\theta : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ such that for every $\mathcal{K}$-cocomplete ∞-category $\mathcal{E}$, precomposition with θ induces an equivalence:*

$$\mathrm{Fun}_{\mathcal{K}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow[\simeq]{-\circ\theta} \mathrm{Bicc}_{\mathcal{K}}(\mathcal{C} \times \mathcal{D}, \mathcal{E})^{\simeq},$$

and the internal hom from $\mathcal{C}$ to $\mathcal{D}$ is given by $\mathrm{Fun}_{\mathcal{K}}(\mathcal{C}, \mathcal{D})$.

Moreover, for every inclusion $\mathcal{K} \subseteq \mathcal{K}'$ of classes of simplicial sets, the forgetful functor $\infty\mathrm{Cat}(\mathcal{K}') \hookrightarrow \infty\mathrm{Cat}(\mathcal{K})$ is lax symmetric monoidal and admits a (strongly) symmetric monoidal left adjoint denoted

$$\infty\mathrm{Cat}(\mathcal{K}) \underset{\xleftarrow{\hspace{1.5cm}}}{\overset{\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}}{\xrightarrow{\hspace{1.5cm}}}} \underset{\perp}{} \infty\mathrm{Cat}(\mathcal{K}')$$

and called the relative cocompletion *of $\mathcal{C}$ from $\mathcal{K}$ to $\mathcal{K}'$. When $\mathcal{K} = \varnothing$, we also write $\mathcal{P}^{\mathcal{K}'} := \mathcal{P}_{\varnothing}^{\mathcal{K}'}$.*

To somewhat explain how one goes about proving such a result, it relies on the following construction and result:

Definition 7.2.4 ([Lur17, Notation 4.8.1.2])**.** Let $\infty\,\mathrm{Cat}^{\times} \to \mathrm{N}\,(\mathrm{Fin}_*)$ denote the cartesian symmetric monoidal structure on $\infty\,\mathrm{Cat}$, and P denote the very large poset of sets of small simplicial sets, ordered under inclusion. The objects of $\infty\,\mathrm{Cat}^{\times}$ can be identified with tuples $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ of ∞-categories, and the morphisms $(\mathcal{C}_1, \ldots, \mathcal{C}_m) \to (\mathcal{D}, \ldots, \mathcal{D}_n)$ over a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathrm{N}\,(\mathrm{Fin}_*)$ with tuples of functors

$$f_j : \prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j.$$

Write $\mathcal{M} \subseteq \infty\,\mathrm{Cat}^{\times} \times \mathrm{N}\,(P)$ for the subcategory spanned by the pairs $((\mathcal{C}_1, \ldots, \mathcal{C}_n), \mathcal{K})$ where every $\mathcal{C}_i$ is a $\mathcal{K}$-cocomplete ∞-category, and the morphisms $((\mathcal{C}_1, \ldots, \mathcal{C}_m), \mathcal{K}) \to ((\mathcal{D}_1, \ldots, \mathcal{C}_n), \mathcal{K}')$ whose underlying functors f_j are $\mathcal{K}$-cocontinuous for every $1 \leq i \leq j$ (which makes sense since $\mathcal{K} \subseteq \mathcal{K}'$).

Fact 7.2.5 ([Lur17, Prop 4.8.1.3])**.** *The forgetful functor $p : \mathcal{M} \to \mathrm{N}\,(\mathrm{Fin}_*) \times \mathrm{N}\,(P)$ is a cocartesian fibration.*

Proof idea. First consider the case of a morphism $f : (\langle 1 \rangle, \mathcal{K}) \to (\langle 1 \rangle, \mathcal{K}')$ in $\mathrm{N}\,(\mathrm{Fin}_*) \times \mathrm{N}\,(P)$ whose underlying morphism in $\mathrm{N}\,(\mathrm{Fin}_*)$ is $\mathrm{id}_{\langle 1 \rangle}$. That this morphism admits a cocartesian lift means, in essence[1], that every $\mathcal{K}$-cocomplete ∞-category $\mathcal{C}$ admits a relative cocompletion from $\mathcal{K}$ to $\mathcal{K}'$, i.e. a $\mathcal{K}$-cocontinuous functor $F : \mathcal{C} \to \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}\mathcal{C}$ that's universal among $\mathcal{K}$-cocontinuous functors with codomain a $\mathcal{K}'$-cocomplete ∞-category.

The existence of such a relative cocompletion is shown in [Lur09, Prop 5.3.6.2]: when $\mathcal{K} = \varnothing$, the idea is quite simple: by fact 6.6.32, the ∞-category

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}\,(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{\mathcal{U}_1})$$

is the cocompletion of $\mathcal{C}$ with respect to all $\mathcal{U}_1$-small colimits, so we can take

$$\mathcal{P}_{\varnothing}^{\mathrm{sSet}\,\mathcal{U}_1} := \mathcal{P}(\mathcal{C}).$$

For general $\mathcal{K}'$, we can take $\mathcal{P}_{\varnothing}^{\mathcal{K}'}$ to be the smallest full subcategory of $\mathcal{P}(\mathcal{C})$ containing the representable presheaves and stable under K-indexed colimits for $K \in \mathcal{K}'$. For $\mathcal{K} \neq \varnothing$ Lurie considers instead the *localization* $S^{-1}\mathcal{P}(\mathcal{C})$ where S is a suitable class of morphisms in $\mathcal{P}(\mathcal{C})$. We won't go into the detail of localizations of ∞-categories here, referring the interested reader to [Lur09, Section 5.2.7] and [Lur18, Tag 01M4]. The main takeaway here is that $S^{-1}\mathcal{P}(\mathcal{C})$ is a full subcategory of $\mathcal{P}(\mathcal{C})$ whose inclusion functor admits a right adjoint, such that the composition

$$\mathcal{C} \xrightarrow{\,\sharp_{\mathcal{C}}\,} \mathcal{P}(\mathcal{C}) \to S^{-1}\mathcal{P}(\mathcal{C})$$

is $\mathcal{K}$-cocontinuous, so that we can define $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}\mathcal{C}$ to be the smallest $\mathcal{K}'$-cocomplete full subcategory of $S^{-1}\mathcal{P}(\mathcal{C})$ containing the essential image of $\mathcal{C}$.

The case of cocartesian lifts for more general morphisms $(\langle m \rangle, \mathcal{K}) \to (\langle n \rangle, \mathcal{K}')$ is a variation on the same idea. $\square$

Proof idea for fact 7.2.3. Write $p : \mathcal{M} \to \mathrm{N}\,(\mathrm{Fin}_*) \times \mathrm{N}\,(P)$ for the cocartesian fibration of fact 7.2.5. For every $\mathcal{K} \in P$, write

$$\mathcal{M}_{\mathcal{K}} := \mathcal{M} \times_{\mathrm{N}\,(\mathrm{Fin}_*) \times \mathrm{N}\,(P)} \{\mathcal{K}\}.$$

---

[1] Formally speaking, we are stating here the existence of a *locally cocartesian lift*, a slightly weaker condition, which then relies on a compatibility with composition to show they are indeed cocartesian lifts.

Since cocartesian fibrations are stable under pullback, the induced functor $q : \mathcal{M}_{\mathcal{K}} \to \mathrm{N}\,(\mathrm{Fin}_*)$ is a cocartesian fibration. Unfolding the construction of $\mathcal{M}$, we can moreover see that

$$(\mathcal{M}_{\mathcal{K}})_{\langle 1 \rangle} \simeq \infty\,\mathrm{Cat}(\mathcal{K}),$$

and that q induces equivalences

$$(\mathcal{M}_{\mathcal{K}})_{\langle n \rangle} \simeq \infty\,\mathrm{Cat}(\mathcal{K})^n,$$

so $\mathcal{M}_{\mathcal{K}}$ is a symmetric monoidal ∞-category with underlying ∞-category $\infty\,\mathrm{Cat}(\mathcal{K})$, and we can define $\infty\,\mathrm{Cat}(\mathcal{K})^{\otimes} := \mathcal{M}_{\mathcal{K}}$.

The cocartesian fibration p corresponds under straightening (fact 6.4.27) to a functor

$$\mathrm{N}\,(P) \to \mathrm{CMon}\,(\infty\,\mathrm{Cat}^{\times}_{\mathcal{U}_1})$$

whose action on the inclusions $\mathcal{K} \to \mathcal{K}'$ is given by the relative cocompletion $\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}$ described in the proof of fact 7.2.5, so the relative cocompletion is a symmetric monoidal functor. The universal property of the functor

$$\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \infty\,\mathrm{Cat}(\mathcal{K}) \to \infty\,\mathrm{Cat}(\mathcal{K}')$$

shows that it is left adjoint to the forgetful functor

$$\infty\,\mathrm{Cat}(\mathcal{K}') \hookrightarrow \infty\,\mathrm{Cat}(\mathcal{K}),$$

so by fact 6.11.13 this forgetful functor is lax symmetric monoidal (although this can also be noticed directly by inspecting the cocartesian fibration p).

The equivalence

$$\mathrm{Fun}_{\mathcal{K}}\,(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow[\simeq]{-\circ\theta} \mathrm{Bicc}_{\mathcal{K}}\,(\mathcal{C} \times \mathcal{D}, \mathcal{E})^{\simeq}.$$

follows from the definition of the cocartesian fibration $p : \mathcal{M} \to \mathrm{N}\,(\mathrm{Fin}_*) \times \mathrm{N}\,(P)$ together with the definition of the tensor product as the covariant transport functor over the morphism $\langle 2 \rangle \to \langle 1 \rangle$ given by $1, 2 \mapsto 1$.

Finally, the internal hom from $\mathcal{C}$ to $\mathcal{D}$: there is for every $\mathcal{E}$ a canonical equivalence

$$\mathrm{Bicc}_{\mathcal{K}}\,(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}_{\mathcal{K}}\,(\mathcal{C}, \mathrm{Fun}_{\mathcal{K}}\,(\mathcal{D}, \mathcal{E}))$$

because of a Fubini-like theorem for ∞-categorical colimits [Lur09, Lem 5.5.2.3], thus inducing a homotopy equivalence

$$\mathrm{Hom}_{\infty\,\mathrm{Cat}(\mathcal{K})}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Hom}_{\infty\,\mathrm{Cat}(\mathcal{K})}(\mathcal{C}, \mathrm{Fun}_{\mathcal{K}}\,(\mathcal{D}, \mathcal{E})).$$

$\square$

Remark 7.2.6. Fact 7.2.3 can be used to obtain a similar result for 1-categories, or more generally $(n, 1)$-categories by *localizing* the monoidal structures [Lur17, Prop 2.2.1.9]. In particular, this can be used to give a proof of fact 3.11.9.

7.2.2 The Scott exponentials

We can now use fact 7.2.3 to build ∞-categorical linear/non-linear adjunctions similar to the ones of section 3.11.2, using the following fact:

Fact 7.2.7 ([Lur17, Rmk 4.8.1.5])**.** *Let $\mathcal{K}$ be a class of* sifted *simplicial sets and let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ ∞-categories $\mathcal{K}$-cocomplete. Then a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is $\mathcal{K}$-cocontinuous if and only if it is $\mathcal{K}$-bicontinuous.*

Corollary 7.2.8. *If $\mathcal{K}$ is a class of sifted simplicial sets, then the symmetric monoidal ∞-category $\infty\,\mathrm{Cat}(\mathcal{K})^{\otimes}$ is cartesian symmetric monoidal.*

Proof idea for fact 7.2.7. First suppose that F is $\mathcal{K}$-bicontinuous, let $K \in \mathcal{K}$ and $\langle u, v \rangle : K \to \mathcal{C} \times \mathcal{D}$ be a diagram. By definition of K being sifted, the diagonal morphism $\Delta : K \to K \times K$ is cofinal, so by fact 6.6.18 the colimits of the diagrams

$$\langle u, v \rangle : K \to \mathcal{C} \times \mathcal{D} \quad \text{and} \quad u \times v : K \times K \to \mathcal{C} \times \mathcal{D}$$

are canonically isomorphic. Informally, we have the following chain of isomorphisms in $\mathcal{E}$:

$$
\begin{aligned}
F(\mathrm{colim}\,\langle u, v \rangle) &= F(\mathrm{colim}_{\,k \in K}(u(k), v(k))) \\
&\simeq F\big(\mathrm{colim}_{\,(i,j) \in K \times K}(u(i), v(j))\big) &&\text{(K sifted)} \\
&\simeq F\Big(\mathrm{colim}_{\,i \in K}\big(\mathrm{colim}_{\,j \in K}(u(i), v(j))\big)\Big) \\
&\simeq \mathrm{colim}_{\,i \in K}\Big(F\big(\mathrm{colim}_{\,j \in J}(u(i), v(j))\big)\Big) &&\text{(F is $\mathcal{K}$-bicontinuous)} \\
&\simeq \mathrm{colim}_{\,i \in K}\mathrm{colim}_{\,j \in K}F\big(u(i), v(j)\big) &&\text{(F is $\mathcal{K}$-bicontinuous)} \\
&\simeq \mathrm{colim}_{\,(i,j) \in K \times K}F\big(u(i), v(j)\big) \\
&\simeq \mathrm{colim}_{\,k \in K}F(u(k), v(k)) &&\text{(K sifted)} \\
&= \mathrm{colim}\,(F \circ \langle u, v \rangle)
\end{aligned}
$$

so F preserves K-indexed colimits (note that this is argument is not entirely formal because of remark 6.7.14, for a more formal proof see [Lur09, Prop 5.5.8.6]).

Conversely suppose that $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is $\mathcal{K}$-cocontinuous. Let $K \in \mathcal{K}$, $u : K \to \mathcal{C}$ be a diagram and d an object of $\mathcal{D}$. Since K is sifted, it is weakly contractible ([Lur09, Prop 5.5.8.7]). The colimit of a constant diagram $v : K \to \{d\} \hookrightarrow \mathcal{D}$ indexed by the weakly contractible K is d [Lur09, Cor 4.4.4.10] (this is the ∞-categorical variant of the statement that the 1-categorical colimit of a constant diagram over a *connected* category is the constant object) In other words, $\mathrm{colim}_{\,K}d \simeq d$. Thus we have a chain of isomorphisms

$$
\begin{aligned}
F((\mathrm{colim}\,u), d) &\simeq F(\mathrm{colim}_{\,k \in K}u(k), \mathrm{colim}_{\,k \in K}d) \\
&\simeq F(\mathrm{colim}_{\,k \in K}(u(k), d)) &&\text{(colimits in $\mathcal{C} \times \mathcal{D}$ computed pointwise: fact 6.6.30)} \\
&\simeq \mathrm{colim}_{\,k \in K}F(u(k), d) &&\text{(F $\mathcal{K}$-cocontinuous)} \\
&= \mathrm{colim}\,(F(-, d) \circ u)
\end{aligned}
$$

so $F(-, d)$ preserves K-indexed colimits, and a symmetrical argument shows that $F(c, -)$ preserves K-indexed colimits, so F is indeed $\mathcal{K}$-bicontinuous, which concludes the proof. $\qquad \square$

Combining fact 7.2.3 and corollary 7.2.8, we get the following:

Theorem 7.2.9. *Let $\mathcal{K} \subseteq \mathcal{K}'$ be two classes of simplicial sets such that every $K \in \mathcal{K}$ is sifted. Then there is a linear/non-linear adjunction*

$$
\infty\,\mathrm{Cat}(\mathcal{K})^{\times} \quad \underset{\longleftarrow}{\overset{\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}}{\longrightarrow}} \quad \infty\,\mathrm{Cat}(\mathcal{K}')^{\otimes}
$$

In particular, writing $\infty\,\mathrm{Cat}_{\mathrm{cc}}$, (resp. $\infty\,\mathrm{Cat}_{\mathrm{filtr}}$, $\infty\,\mathrm{Cat}_{\mathrm{sift}}$) for $\infty\,\mathrm{Cat}(\mathcal{K})$ in the case where $\mathcal{K}$ is the class of all small simplicial sets (resp. filtered simplicial sets, sifted simplicial sets), we can specialize theorem 7.2.9 to get the following ∞-categorical variant of fact 3.11.26:

Corollary 7.2.10. *There are linear/non-linear adjunctions*

$$\infty\,\mathrm{Cat}_{\mathrm{filtr}}^{\times} \underset{\perp}{\overset{\mathcal{P}_{\mathrm{filtr}}^{\mathrm{all}}}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathrm{cc}}^{\otimes}$$

$$\infty\,\mathrm{Cat}_{\mathrm{sift}}^{\times} \underset{\perp}{\overset{\mathcal{P}_{\mathrm{sift}}^{\mathrm{all}}}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathrm{cc}}^{\otimes}$$

Proof. This follows from theorem 7.2.9, using the fact that every filtered simplicial set is sifted (fact 6.6.17). □

As promised in section 3.11.2, we can even extend this to a chain of monoidal adjunctions:

Corollary 7.2.11. *There is a chain of symmetric monoidal left adjoints:*

$$\mathcal{S}_{\mathcal{U}_1}^{\times} \underset{(-)^{\simeq}}{\overset{\perp}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathcal{U}_1}^{\times} \underset{\perp}{\overset{\mathcal{P}^{\mathrm{filtr}}}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathrm{filtr}}^{\times} \underset{\perp}{\overset{\mathcal{P}_{\mathrm{filtr}}^{\mathrm{sift}}}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathrm{sift}}^{\times} \underset{\perp}{\overset{\mathcal{P}_{\mathrm{sift}}^{\mathrm{all}}}{\rightleftarrows}} \infty\,\mathrm{Cat}_{\mathrm{cc}}^{\otimes}$$

Where all the symmetric monoidal structures involved are cartesian except for $\infty\,\mathrm{Cat}_{\mathrm{cc}}^{\otimes}$.

Proof. The only thing left to check is that the inclusion functor $\mathcal{S}^{\times} \hookrightarrow \infty\,\mathrm{Cat}^{\times}$ is symmetric monoidal, this follows from the fact that it preserves finite products, and that a functor between cartesian symmetric monoidal ∞-categories is symmetric monoidal if and only if it preserves finite products [Lur17, Cor 2.4.1.8]. □

In fact 3.11.26, we also claimed that the induced comonads on $\infty\,\mathrm{Cat}_{\mathrm{cc}}$ could be explicitly described when restricted to *profunctors*. The way this works is the same in essence as it was at the posetal level for domains and suplattices (section 3.9):

Definition 7.2.12. Let $\mathcal{C}$ and $\mathcal{D}$ be small ∞-categories. A *profunctor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$.

We write $\infty\,\mathrm{Prof}$ for the full subcategory of $\infty\,\mathrm{Cat}_{\mathrm{cc}}$ spanned the ∞-categories of the form $\mathcal{P}(\mathcal{C})$ for $\mathcal{C}$ a small ∞-category.

Note that by fact 6.6.32, we have a canonical homotopy equivalence

$$\mathrm{Hom}_{\infty\,\mathrm{Prof}}(\mathcal{P}(\mathcal{C}), \mathcal{P}(\mathcal{D})) \simeq \mathrm{Fun}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}}, \mathcal{S}),$$

so that $\infty\,\mathrm{Prof}$ may indeed be thought as an ∞-category of small ∞-categories and profunctors between them.

Remark 7.2.13. Formally speaking, the above intuition is slightly incorrect because the underlying ∞-groupoids $\infty\,\mathrm{Prof}^{\simeq}$ and $\infty\,\mathrm{Cat}^{\simeq}$ are not actually equivalent: two non-equivalent ∞-categories may have equivalent ∞-categories of presheaves. In general, when $\mathcal{C}$ and $\mathcal{D}$ are small ∞-categories, $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}(\mathcal{D})$ are equivalent if and only if $\mathcal{C}$ and $\mathcal{D}$ have the same *idempotent completion* (see [Lur18, Tag 040X]). In particular, it might be better to think of $\infty\,\mathrm{Prof}$ as the ∞-category of small *idempotent-complete* ∞-categories and profunctors between them.

Remark 7.2.14. Here when $\mathcal{C}$ is a small ∞-category, we write $\mathcal{P}(\mathcal{C})$ for the ∞-category of small presheaves on $\mathcal{C}$, which is equivalent to $\mathcal{P}_{\varnothing}^{\text{all}}$ that was defined as the smallest cocomplete full subcategory of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S}_{\mathcal{U}_1})$ containing the representable presheaves [Lur09, Ex 5.3.6.6]. This equivalence also holds when $\mathcal{C}$ is locally small, but when $\mathcal{C}$ is large then in general $\mathcal{P}(\mathcal{C})$ is not the free cocompletion of $\mathcal{C}$, as there is not necessarily even a way to embed $\mathcal{C}$ in $\mathcal{P}(\mathcal{C})$.

Notation 7.2.15. Write fin for the class of finite simplicial sets, and $\sqcup$ for the class of discrete finite simplicial sets.

The following is an ∞-categorical variant to fact 3.9.16:

Fact 7.2.16 ([Lur09, Ex 5.3.6.8, Rmk 5.5.8.16]). *There are natural isomorphisms of functors:*

$$\mathcal{P}^{\text{sift}} \circ \mathcal{P}^{\sqcup} \simeq \mathcal{P}^{\text{all}}$$
$$\mathcal{P}^{\text{filtr}} \circ \mathcal{P}^{\text{fin}} \simeq \mathcal{P}^{\text{all}}$$

In particular, when $\mathcal{C}$ is a small ∞-category, we have natural equivalences:

$$\mathcal{P}(\mathcal{C}) \simeq \mathcal{P}^{\text{sift}}(\mathcal{P}^{\sqcup}(\mathcal{C})) \simeq \mathcal{P}^{\text{filtr}}(\mathcal{P}^{\text{fin}}(\mathcal{C}))$$

As a corollary, we get the ∞-categorical version of the last statement from fact 3.11.26:

Corollary 7.2.17. *When restricted to $\infty\,\text{Prof}$, the comonads induced by the linear/non-linear adjunctions*

$$\infty\,\text{Cat}_{\text{sift}} \underset{\perp}{\overset{\mathcal{P}_{\text{sift}}^{\text{all}}}{\rightleftarrows}} \infty\,\text{Cat}_{\text{cc}}{}^{\otimes} \qquad\qquad \infty\,\text{Cat}_{\text{filtr}} \underset{\perp}{\overset{\mathcal{P}_{\text{filtr}}^{\text{all}}}{\rightleftarrows}} \infty\,\text{Cat}_{\text{cc}}{}^{\otimes}$$

respectively act as follows on objects:

$$\mathcal{P}(\mathcal{C}) \mapsto \mathcal{P}(\mathcal{P}^{\sqcup}(\mathcal{C}))$$
$$\mathcal{P}(\mathcal{C}) \mapsto \mathcal{P}(\mathcal{P}^{\text{fin}}(\mathcal{C}))$$

Proof. We have

$$\begin{aligned}
\mathcal{P}_{\text{sift}}^{\text{all}}(\mathcal{P}(\mathcal{C})) &\simeq \mathcal{P}_{\text{sift}}^{\text{all}}(\mathcal{P}^{\text{sift}}(\mathcal{P}^{\sqcup}(\mathcal{C}))) && \text{(fact 7.2.16)} \\
&\simeq \mathcal{P}^{\text{all}}(\mathcal{P}^{\sqcup}(\mathcal{C})) \\
&\simeq \mathcal{P}(\mathcal{P}^{\sqcup}(\mathcal{C}))
\end{aligned}$$

where the last step is due to the fact that $\mathcal{P}^{\sqcup}(\mathcal{C})$ is still essentially small when $\mathcal{C}$ is small. A similar computation allows us to conclude for $\mathcal{P}^{\text{fin}}$ and $\mathcal{P}^{\text{filtr}}$. $\qquad\square$

7.2.3 The Lafont exponential

We now turn our attention to the free exponential on $\infty\,\text{Prof}$, i.e. to the existence of cofree commutative comonoids on $\infty\,\text{Prof}$, generalizing to the ∞-categorical setting the results we review in section 3.11.1.

Proposition 7.2.18. *The symmetric monoidal structure on $\infty\,\text{Cat}_{\text{cc}}$ restricts to $\infty\,\text{Prof}$. Moreover, we have $\mathcal{P}1 \simeq \mathcal{S}$ and $\mathcal{PC} \otimes \mathcal{PD} \simeq \mathcal{P}(\mathcal{C} \times \mathcal{D})$.*

Proof. Writing $\infty\operatorname{Prof}^\otimes \subseteq \infty\operatorname{Cat}_{cc}^\otimes$ for the full subcategory spanned by the tuples $(\mathcal{P}(\mathcal{C}_1), \dots, \mathcal{P}(\mathcal{C}_n))$ of presheaf ∞-categories on small ∞-categories. We need to show that the composite functor $\infty\operatorname{Prof}^\otimes \to \mathrm{N}(\operatorname{Fin}_*)$ is still a symmetric monoidal ∞-category. Unfolding definitions, it suffices to show that $\infty\operatorname{Prof}$ contains the unit of $\infty\operatorname{Cat}_{cc}^\otimes$ and is closed under tensor products in $\infty\operatorname{Cat}_{cc}^\otimes$ [Lur17, Prop 2.2.1.1].

The free cocompletion functor $\mathcal{P}^{\mathrm{all}}(-) : \operatorname{Cat}_{\mathcal{U}_1} \to \operatorname{Cat}_{cc}$ is a symmetric monoidal functor, and the monoidal structure on $\infty\operatorname{Cat}$ is cartesian by corollary 7.2.8. Since $\mathcal{P}^{\mathrm{all}}(\mathcal{C}) \simeq \mathcal{P}(\mathcal{C})$ whenever $\mathcal{C}$ is a small ∞-category, we get isomorphisms $\mathcal{S} \simeq \mathcal{P}(1)$ and $\mathcal{P}(\mathcal{C}) \otimes \mathcal{P}(\mathcal{D}) \simeq \mathcal{P}(\mathcal{C} \times \mathcal{D})$. It remains to show that the unit and the tensor product on $\infty\operatorname{Cat}^\times$ preserve smallness: the terminal ∞-category is clearly small and the cartesian product of small ∞-categories is still a small ∞-category. $\square$

To prove that $\infty\operatorname{Prof}$ admits cofree commutative comonoids, we will rely on proposition 7.1.17, first showing that it is $*$-autonomous by way of it being compact closed (by using corollary 6.12.13), and then show that it admits free commutative monoids via the techniques we reviewed in section 6.11.2.

Fact 7.2.19 ([AL18, Thm 5.6])**.** *The symmetric monoidal ∞-category $\infty\operatorname{Prof}^\otimes$ is compact closed, i.e. all its objects are dualizable (definition 6.12.8).*

Proof idea. Let $\mathcal{C}$ be a small ∞-category. We show that $\mathcal{P}(\mathcal{C}^{\mathrm{op}})$ is a dual of $\mathcal{P}(\mathcal{C})$ in $\infty\operatorname{Cat}_{cc}$, and hence in $\infty\operatorname{Prof}$. We need to specify cocontinuous functors

$$c : \mathcal{S} \to \mathcal{P}(\mathcal{C}) \otimes \mathcal{P}(\mathcal{C}^{\mathrm{op}}) \qquad e : \mathcal{P}(\mathcal{C}^{\mathrm{op}}) \otimes \mathcal{P}(\mathcal{C}) \to \mathcal{S}.$$

There is a chain of equivalences:

$$\operatorname{Fun}_{cc}(\mathcal{P}(\mathcal{C}^{\mathrm{op}}) \otimes \mathcal{P}(\mathcal{C}), \mathcal{S}) \simeq \operatorname{Fun}_{cc}(\mathcal{P}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}), \mathcal{S}) \qquad \text{(by proposition 7.2.18)}$$
$$\simeq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S}) \qquad \text{(by fact 6.6.32)}$$

Under the inverse of this equivalence, we can take c to be the hom functor $\operatorname{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$.

We also have a chain of equivalences:

$$\operatorname{Fun}_{cc}(\mathcal{S}, \mathcal{P}(\mathcal{C}) \otimes \mathcal{P}(\mathcal{C}^{\mathrm{op}})) \simeq \operatorname{Fun}_{cc}(\mathcal{P}(1), \mathcal{P}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}})) \qquad \text{(by proposition 7.2.18)}$$
$$\simeq \operatorname{Fun}(1, \mathcal{P}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}})) \qquad \text{(by fact 6.6.32)}$$
$$\simeq \mathcal{P}(\mathcal{C} \times \mathcal{C}^{\mathrm{op}})$$
$$= \operatorname{Fun}((\mathcal{C} \times \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{S})$$
$$\simeq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$$

under which we can also take e to be the hom functor $\operatorname{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$.

Unfolding these inverses, c and e are expressed using ∞-categorical *left Kan extensions* or *coends*, and in the end the commutativity of both triangles in definition 6.12.8 follows from the ∞-categorical coend-formula

$$\int^{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(c, d) \times F(d) \simeq F(d)$$

for every $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ and $c \in \mathcal{C}_0$ (see [AL18, Thm 5.6] for more details). $\square$

Applying corollary 6.12.13 to fact 7.2.19, we get:

Corollary 7.2.20. *The unit* $\mathcal{S} = \mathcal{P}(1)$ *of* $\infty\,\mathrm{Prof}^{\otimes}$ *is a dualizing object, so* $(\infty\,\mathrm{Prof}^{\otimes}, \mathcal{S})$ *is a* ∗-*autonomous* ∞-*category.*

Remark 7.2.21. In [HM25], we gave a direct proof of the ∗-autonomy of ∞ Prof without relying on the existence of duals. This proof was more computational, but the main ingredient also ended up being a similar coend computation as in the proof we just presented.

We now turn to the existence of free commutative monoids in ∞ Prof.

Theorem 7.2.22. *The* ∞-*category* $\infty\,\mathrm{Cat}_{cc}$ *admits small colimits, and for every cocomplete* ∞-*category* $\mathcal{C}$ *the tensor product* $\mathcal{C} \otimes - : \infty\,\mathrm{Cat}_{cc} \to \infty\,\mathrm{Cat}_{cc}$ *preserves small colimits.*

This follows from the following stronger statement:

Theorem 7.2.23. *Given a* (*not necessarily small*) *collection* $\mathcal{K}$ *of small simplicial sets, the* ∞-*category* $\infty\,\mathrm{Cat}(\mathcal{K})$ *admits small colimits, and for every* $\mathcal{K}$-*cocomplete* ∞-*category* $\mathcal{C}$ *the tensor product* $\mathcal{C} \otimes - :$ $\infty\,\mathrm{Cat}_{cc} \to \infty\,\mathrm{Cat}_{cc}$ *preserves small colimits.*

Proof. Since $\infty\,\mathrm{Cat}(\mathcal{K})$ is closed symmetric monoidal (fact 7.2.3), the tensor product functor $\mathcal{C} \otimes - : \infty\,\mathrm{Cat}(\mathcal{K}) \to \infty\,\mathrm{Cat}(\mathcal{K})$ is a left adjoint, so it preserves any colimits that may exist in $\infty\,\mathrm{Cat}(\mathcal{K})$ (fact 6.7.18).

We are left with proving that $\infty\,\mathrm{Cat}(\mathcal{K})$ admits small colimits. Proposition 4.8.4.2 in [Lur17] states that $\infty\,\mathrm{Cat}(\mathcal{K})$ is presentable for every *small* collection $\mathcal{K}$ of small simplicial sets. Without the smallness assumption on $\mathcal{K}$, presentability fails because $\infty\,\mathrm{Cat}(\mathcal{K})$ is no longer generally accessible, but the existence of small colimits still holds, as we now show:

The proof follows closely the one of [Lur17, Prop 4.8.4.2], but keeping track of size issues. Recall that we fixed three Grothendieck universes $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2$, and we defined $\infty\,\mathrm{Cat}(\mathcal{K})$ as a subcategory of $\infty\,\mathrm{Cat}_{\mathcal{U}_1}$, the very large ∞-category of large ∞-categories (where small means $\mathcal{U}_0$-small, large means $\mathcal{U}_1$-small and very large means $\mathcal{U}_2$-small). We write $\infty\,\mathrm{Cat}_{\mathcal{U}_2}(\mathcal{K})$ for the subcategory of $\infty\,\mathrm{Cat}_{\mathcal{U}_2}$ spanned by very large ∞-categories that are $\mathcal{K}$-cocomplete. Given a diagram $\chi : J \to \infty\,\mathrm{Cat}(\mathcal{K})$, proposition 4.8.4.2 of [Lur17] guarantees that there exists a very large ∞-category $\mathcal{C}$ which is a colimit of this diagram in $\infty\,\mathrm{Cat}_{\mathcal{U}_2}(\mathcal{K})$. We will show that $\mathcal{C}$ is actually (essentially) $\mathcal{U}_1$-small.

The construction of the colimit $\mathcal{C}$ is detailed in [Lur09, Prop 5.3.6.2]: first take a colimit $\mathcal{D}$ of the composition

$$ J \xrightarrow{\chi} \infty\,\mathrm{Cat}(\mathcal{K}) \to \infty\,\mathrm{Cat}. $$

Let $\mathcal{P}_{\mathcal{U}_1}(\mathcal{D}) := \mathrm{Fun}\,(\mathcal{D}^{\mathrm{op}}, \mathcal{S}_{\mathcal{U}_1})$ denote the ∞-category of $\mathcal{U}_1$-small presheaves on $\mathcal{D}$. By definition, $\mathcal{P}_{\mathcal{U}_1}(\mathcal{D})$ is $\mathcal{U}_1$-cocomplete, but the canonical functors $\chi(i) \to \mathcal{P}_{\mathcal{U}_1}(\mathcal{D})$ for $i \in J_0$ need not be $\mathcal{K}$-cocontinuous. Hence one must then take a localization of $\mathcal{P}_{\mathcal{U}_1}(\mathcal{D})$ to "force" these functors to be $\mathcal{K}$-cocontinuous, in other words one can find a full subcategory $\mathbb{D} \subseteq \mathcal{P}_{\mathcal{U}_1}(\mathcal{D})$ such that the

inclusion functor admits a left adjoint $r\colon\ \mathbb{D} \underset{\longrightarrow}{\overset{r}{\underset{\perp}{\longleftarrow}}} \widehat{\mathcal{D}}$ The ∞-category $\mathcal{C}$ is finally obtained by considering the smallest $\mathcal{K}$-cocomplete full subcategory of $\mathbb{D}$ containing the essential image of the composite functor

$$ \mathcal{D} \hookrightarrow \mathcal{P}_{\mathcal{U}_1}(\mathcal{D}) \xrightarrow{r} \mathbb{D}. $$

The ∞-category $\mathcal{P}_{\mathcal{U}_1}(\mathcal{D})$ is locally $\mathcal{U}_1$-small because it is an ∞-category of $\mathcal{U}_1$-small presheaves (remark 6.5.9), so $\mathcal{C}$ is also locally $\mathcal{U}_1$-small (as a full subcategory of a locally small ∞-category). To show it is essentially $\mathcal{U}_1$-small, it suffices by fact 6.4.20 to show that the set of isomorphism classes of its objects is $\mathcal{U}_1$-small.

Write $(\mathcal{C})_0$ for the essential image of the functor $\mathcal{D} \hookrightarrow \widehat{\mathcal{D}} \xrightarrow{r} \mathbb{D}$. Since $\mathcal{D}$ is $\mathcal{U}_1$-small and $\mathbb{D}$ is locally $\mathcal{U}_1$-small, $(\mathcal{C})_0$ is essentially $\mathcal{U}_1$-small. By definition, $\mathcal{C}$ is the smallest $\mathcal{K}$-cocomplete full

subcategory of $\mathbb{D}$ containing $(\mathcal{C})_0$. We describe $\mathcal{C}$ more explicitly as the union of a monotonous sequence $((\mathcal{C})_\alpha)$ of full subcategories of $\mathbb{D}$, where α takes values in all $\mathcal{U}_0$-small ordinals:

- when $\alpha = \alpha' + 1$, define $(\mathcal{C})_\alpha$ to be the full subcategory of $\mathbb{D}$ on objects that are colimits of diagrams $K \to (\mathcal{C})_{\alpha'}$, for $K \in \mathcal{K}$.

- when $\alpha = \lim \alpha_i$, define $(\mathcal{C})_\alpha$ to be the union of the $(\mathcal{C})_{\alpha_i}$.

Write ϵ for the smallest ordinal with the cardinality of $\mathcal{U}_0$.

Each of the $(\mathcal{C})_\alpha$ for $\alpha \leq \epsilon$ is $\mathcal{U}_1$-small:

- $(\mathcal{C})_0$ is $\mathcal{U}_1$-small as has already been observed.

- Assume $(\mathcal{C})_\alpha$ is $\mathcal{U}_1$-small. Since $\mathcal{K}$ consists of $\mathcal{U}_0$-small simplicial sets, it is $\mathcal{U}_1$-small, and for every $K \in \mathcal{K}$, there are a $\mathcal{U}_1$-small amount of diagrams $K \to (\mathcal{C})_\alpha$ because $(\mathcal{C})_\alpha \subseteq \mathbb{D}$ and $\mathbb{D}$ is locally $\mathcal{U}_1$-small. Hence $(\mathcal{C})_{\alpha+1}$ is also $\mathcal{U}_1$-small.

- If $\alpha = \lim \alpha_i$ with $\alpha \leq \epsilon$, then we can assume that for every i, $\alpha_i < \epsilon$ so every α_i is $\mathcal{U}_0$-small, so the set $\{\alpha_i\}$ is $\mathcal{U}_1$-small, and $(\mathcal{C})_\alpha$ is the union of a $\mathcal{U}_1$-small family of $\mathcal{U}_1$-small ∞-categories, so it is itself $\mathcal{U}_1$-small.

In particular, $(\mathcal{C}_\epsilon)$ is $\mathcal{U}_1$-small.

We claim that $\mathcal{C} = (\mathcal{C})_\epsilon$. Clearly, every $(\mathcal{C})_\alpha$ is a full subcategory of $\mathcal{C}$, $(\mathcal{C})_\epsilon \subseteq \mathcal{C}$. It thus suffices to show that $(\mathcal{C})_\epsilon$ is $\mathcal{K}$-cocomplete. Let $\xi : K \to (\mathcal{C})_\epsilon$ be a diagram in $(\mathcal{C})_\epsilon$ with $K \in \mathcal{K}$. Since K is $\mathcal{U}_0$-small, it is a $\mathcal{U}_0$-compact object of $\infty \operatorname{Cat}_{\mathcal{U}_1}$ by example 6.8.4, so ξ factors through $(\mathcal{C})_\alpha$ for some $\alpha < \epsilon$, so the colimit of ξ belongs to $(\mathcal{C})_{\alpha+1}$, and hence to $(\mathcal{C})_\epsilon$, which concludes the proof. □

Lemma 7.2.24. *The ∞-category $\infty \operatorname{Cat}_{cc}$ admits free commutative monoids, where we write $\operatorname{Sym}_{cc}(\mathcal{C})$ for the free commutative monoid on $\mathcal{C}$ in $\infty \operatorname{Cat}_{cc}{}^{\otimes}$. Moreover, for every small ∞-category $\mathcal{C}$, there free commutative monoid is a canonical equivalence*

$$\operatorname{Sym}_{cc}(\mathcal{P}(\mathcal{C})) \simeq \mathcal{P}(\operatorname{Sym}(\mathcal{C})),$$

where $\operatorname{Sym}(\mathcal{C})$ denotes the free commutative monoid on $\mathcal{C}$ in $\infty \operatorname{Cat}$, i.e. the free symmetric monoidal ∞-category on $\mathcal{C}$ of example 6.11.21.

Proof. The existence of free commutative monoids in $\infty \operatorname{Cat}_{cc}$ follows from fact 6.11.20 applied to theorem 7.2.22. As already noted in example 6.11.21, $\infty \operatorname{Cat}$ also admits free commutative monoids because of fact 6.11.20. Moreover, fact 6.11.20 guarantees that the free commutative monoids in $\infty \operatorname{Cat}$ and $\infty \operatorname{Cat}_{cc}$ can be computed using the formula

$$\mathcal{C} \mapsto \bigsqcup_{n \geq 0} \mathcal{C}^{\otimes n} /\!\!/ \mathfrak{S}_n, \tag{7.1}$$

with $\otimes$ being the cartesian structure for $\infty \operatorname{Cat}$ and the symmetric monoidal structure of fact 7.2.3 of $\infty \operatorname{Cat}_{cc}$. The functor $\mathcal{P}^{\text{all}} : \infty \operatorname{Cat}_{\mathcal{U}_1} \to \infty \operatorname{Cat}_{cc}$ preserves small colimits by virtue of being a left adjoint, and the inclusion functor $\infty \operatorname{Cat} = \infty \operatorname{Cat}_{\mathcal{U}_0} \overset{\subseteq}{\to} \infty \operatorname{Cat}_{\mathcal{U}_1}$ preserves countable colimits, so the composite functor $\mathcal{P} : \infty \operatorname{Cat} \to \infty \operatorname{Cat}_{cc}$ preserves countable colimits. Moreover, fact 7.2.3 also states that $\mathcal{P}^{\text{all}}$ is strongly symmetric monoidal, and since the inclusion $\infty \operatorname{Cat}_{\mathcal{U}_0} \subseteq \infty \operatorname{Cat}_{\mathcal{U}_1}$ preserves finite limits, the functor $\mathcal{P}$ can also be extended to a symmetric monoidal functor

$$\mathcal{P} : \infty \operatorname{Cat}^{\times} \to \infty \operatorname{Cat}_{cc}{}^{\otimes}$$

that preserves countable colimits on the underlying ∞-categories. In particular, $\mathcal{P}$ preserves the constructions involved in the formula (7.1), so it preserves free commutative monoids. □

Remark 7.2.25. The equivalence $\mathrm{Sym}_{cc}\,(\mathcal{P}(\mathcal{C})) \simeq \mathcal{P}(\mathrm{Sym}\,(\mathcal{C}))$, and more generally

$$\mathrm{Sym}_{cc}\,(\mathcal{P}^{\mathrm{all}}(\mathcal{C})) \simeq \mathcal{P}^{\mathrm{all}}(\mathrm{Sym}\,(\mathcal{C})),$$

can be made to be natural in $\mathcal{C}$. Indeed, it can be shown that any symmetric monoidal adjunction induces an adjunction between ∞-categories of commutative monoids ([Lur17, Ex 7.3.2.8]), so in particular the symmetric monoidal adjunction

$$\infty\,\mathrm{Cat}_{\mathcal{U}_1}^{\times} \underset{U}{\overset{\mathcal{P}^{\mathrm{all}}}{\underset{\perp}{\rightleftarrows}}} \infty\,\mathrm{Cat}_{cc}^{\otimes}$$

induces an adjunction

$$\mathrm{CMon}\,(\infty\,\mathrm{Cat}_{\mathcal{U}_1}^{\times}) \underset{\mathrm{CMon}\,(U)}{\overset{\mathrm{CMon}\,(\mathcal{P}^{\mathrm{all}})}{\underset{\perp}{\rightleftarrows}}} \mathrm{CMon}\,(\infty\,\mathrm{Cat}_{cc}^{\otimes})$$

which is moreover compatible with the forgetful functors. Since the square of forgetful right adjoints on the right below commutes (up to natural isomorphism), the square of their left adjoints on the left below also commutes (up to natural isomorphism):

$$
\begin{array}{ccc}
\mathrm{CMon}\,(\infty\,\mathrm{Cat}_{\mathcal{U}_1}) & \xrightarrow{\mathrm{CMon}\,(\mathcal{P}^{\mathrm{all}})} & \mathrm{CMon}\,(\infty\,\mathrm{Cat}_{cc}) \\
{\scriptstyle \mathrm{Sym}}\uparrow & & \uparrow{\scriptstyle \mathrm{Sym}_{cc}} \\
\infty\,\mathrm{Cat}_{\mathcal{U}_1} & \xrightarrow{\ \mathcal{P}^{\mathrm{all}}\ } & \infty\,\mathrm{Cat}_{cc}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{CMon}\,(\infty\,\mathrm{Cat}_{\mathcal{U}_1}) & \xleftarrow{\mathrm{CMon}\,(U)} & \mathrm{CMon}\,(\infty\,\mathrm{Cat}_{cc}) \\
\downarrow & & \downarrow \\
\infty\,\mathrm{Cat}_{\mathcal{U}_1} & \xleftarrow{\ U\ } & \infty\,\mathrm{Cat}_{cc}
\end{array}
$$

Precomposing such a natural isomorphism with the inclusion $\infty\,\mathrm{Cat} \subseteq \infty\,\mathrm{Cat}_{\mathcal{U}_1}$, we get a natural isomorphism

$$\mathrm{Sym}_{cc} \circ \mathcal{P} \simeq \mathcal{P} \circ \mathrm{Sym} \ :\ \infty\,\mathrm{Cat} \to \mathrm{CMon}\,(\infty\,\mathrm{Cat}_{cc}^{\otimes})$$

as desired.

Theorem 7.2.26. *The symmetric monoidal ∞-category $\infty\,\mathrm{Prof}$ is a Lafont ∞-category. Moreover, the cofree commutative comonoid on $\mathcal{P}(\mathcal{C})$ is given by $\mathcal{P}(\mathrm{Sym}\,(\mathcal{C}))$.*

Proof of theorem 7.2.26. Since $\infty\,\mathrm{Cat}_{cc}^{\otimes}$ admits free commutative monoids and $\mathrm{Sym}_{cc}\,(\mathcal{P}(\mathcal{C})) \simeq \mathcal{P}(\mathrm{Sym}\,(\mathcal{C}))$, the adjunction

$$\mathrm{CMon}\,(\infty\,\mathrm{Cat}_{cc}^{\otimes}) \underset{\mathrm{Sym}_{cc}}{\overset{\perp}{\rightleftarrows}} \infty\,\mathrm{Cat}_{cc}$$

can be restricted to an adjunction

$$\mathrm{CMon}\,(\infty\,\mathrm{Prof}^{\otimes}) \underset{\mathrm{Sym}_{cc}}{\overset{\perp}{\rightleftarrows}} \infty\,\mathrm{Prof}$$

so in particular $\infty\,\mathrm{Prof}$ admits free commutative monoids. Moreover, $\infty\,\mathrm{Prof}$ is ∗-autonomous by corollary 7.2.20, so by proposition 7.1.17 $\infty\,\mathrm{Prof}$ admits cofree commutative comonoids, and they are computed as

$$\mathcal{P}(\mathcal{C}) \mapsto \mathrm{Sym}_{cc}\,(\mathcal{P}(\mathcal{C})^{\vee})^{\vee}.$$

Since $\mathcal{P}(\mathcal{C})^\vee \simeq \mathcal{P}(\mathcal{C}^{\mathrm{op}})$, we have

$$\mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{P}(\mathcal{C})^\vee\right)^\vee \simeq \mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{P}(\mathcal{C}^{\mathrm{op}})\right)^\vee$$
$$\simeq \mathcal{P}(\mathrm{Sym}\left(\mathcal{C}^{\mathrm{op}}\right))^\vee$$
$$\simeq \mathcal{P}(\mathrm{Sym}\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}})$$

Unfolding the expression of $\mathrm{Sym}\left(-\right)$ as a coproduct of homotopy quotients, we moreover find that $\mathrm{Sym}\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}} \simeq \mathrm{Sym}\left(\mathcal{C}\right)$, so in the end the cofree commutative comonoid on $\mathcal{P}(\mathcal{C})$ in $\infty\,\mathrm{Prof}$ is indeed given by $\mathcal{P}(\mathrm{Sym}\left(\mathcal{C}\right))$, which concludes the proof. $\qquad\square$

The model of theorem 7.2.26 is an ∞-categorical variant of the model of generalized species ([FGH24]) that we reviewed in section 3.11.1. The Kleisli morphisms $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ for the Lafont comonad on $\infty\,\mathrm{Prof}$ from theorem 7.2.26 correspond to functors

$$\mathrm{Sym}\left(\mathcal{C}\right) \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S},$$

so we can really think of them as an ∞-categorical variant of generalized species. While the approach of [FGH24] to building the bicategorical model of profunctors and generalized species used the intensional perspective on profunctors, ours was more extensional (see section 3.8), directly defining to be a full subcategory of $\infty\,\mathrm{Cat}_{\mathrm{cc}}$. Our approach has the advantage that it lends itself nicely to some further generalizations: if suffices to find intermediary subcategories $\infty\,\mathrm{Prof} \subseteq \mathcal{C} \subseteq \infty\,\mathrm{Cat}_{\mathrm{cc}}$ that are closed under the formation of cofree commutative comonoids to get bigger models of linear logic.

Example 7.2.27. For instance, theorem 5.6 in [AL18] states not only that presheaf ∞-categories are dualizable in $\infty\,\mathrm{Cat}_{\mathrm{cc}}$, but so do their *retracts*. In other words, if $\mathcal{D}$ is a cocomplete ∞-category such that there exists cocontinuous functors $F : \mathcal{D} \to \mathcal{P}(\mathcal{C})$ and $G : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ such that $G \circ F \simeq \mathrm{id}_{\mathcal{D}}$ for $\mathcal{C}$ a small ∞-category, then $\mathcal{D}$ is dualizable $\infty\,\mathrm{Cat}_{\mathrm{cc}}$. The full subcategory $\mathrm{RPsh} \subseteq \infty\,\mathrm{Cat}_{\mathrm{cc}}$ spanned by ∞-categories of the form $\mathcal{P}(\mathcal{C})$ and their retracts is thus compact closed. Moreover, since the condition of $\mathcal{D}$ being a retract is purely diagrammatic, it is preserved by the tensor product, so the symmetric monoidal structure on $\infty\,\mathrm{Cat}_{\mathrm{cc}}$ restricts to RPsh. Similarly, RPsh is stable under the functor $\mathrm{Sym}_{\mathrm{cc}}$: in other words, if $\mathcal{D}$ is a retract of $\mathcal{P}(\mathcal{C})$, then $\mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{D}\right)$ is a retract of $\mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{P}(\mathcal{C})\right)$. Since $\mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{P}(\mathcal{C})\right) \simeq \mathcal{P}(\mathrm{Sym}\left(\mathcal{C}\right))$, this entails that $\mathrm{Sym}_{\mathrm{cc}}\left(\mathcal{D}\right)$ is also a retract of a presheaf ∞-category. Applying proposition 7.1.17, we get that $\mathrm{RPsh}^\otimes$ is a Lafont ∞-category.

Example 7.2.28. Going even further, Anel and Lejay actually state that the presheaf ∞-categories and their retracts are precisely *the* dualizable objects of the ∞-category $\mathrm{Pr}^{\mathrm{L}} \subseteq \infty\,\mathrm{Cat}_{\mathrm{cc}}$ defined in fact 6.8.17. Indeed, Lurie has shown that the symmetric monoidal structure on $\infty\,\mathrm{Cat}_{\mathrm{cc}}$ restricts to Pr^{L}: the tensor product of presentable ∞-categories is still presentable. Recall from fact 6.8.18 and corollary 6.8.19 that Pr^{L} admits small limits and small colimits, the former being computed as limits in $\infty\,\mathrm{Cat}$, and the latter being computed as limits in $\infty\,\mathrm{Cat}$ after composing with the contravariant equivalence $\mathrm{Pr}^{\mathrm{L}} \simeq \mathrm{Pr}^{\mathrm{R}\,\mathrm{op}}$. In particular, if $u : X \to \mathrm{Pr}^{\mathrm{L}}$ is a diagram indexed by an ∞-groupoid X, u factors through $\mathrm{Pr}^{\mathrm{L}\,\simeq}$, so its limit in $\infty\,\mathrm{Cat}$ and the limit of the composite

$$X^{\mathrm{op}} \xrightarrow{u} \mathrm{Pr}^{\mathrm{L}\,\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}} \to \infty\,\mathrm{Cat}$$

are canonically equivalent. In particular, we get a natural isomorphism of functors

$$\lim{}_X \simeq \mathrm{colim}{}_X \;:\; \mathrm{Fun}\left(X, \mathrm{Pr}^{\mathrm{L}}\right) \to \mathrm{Pr}^{\mathrm{L}}.$$

Since the tensor product $\mathcal{C} \otimes -$ on Pr^{L} preserves small colimits, it must also *commute* with small limits. As noted in remark 6.7.14, it is not yet known in the ∞-categorical context if this is enough to guarantee that $\mathcal{C} \otimes -$ preserves small limits, but it is reasonable to expect so. Admitting this result, we can apply a statement dual to fact 6.11.19 to show that Pr^{L} admits cofree commutative comonoids, vastly extending the model of $\infty \mathrm{Prof}$. Note in particular that Pr^{L} is far from being $*$-autonomous.

Remark 7.2.29. The fact that limits and colimits of diagrams in Pr^{L} indexed by ∞-groupoids coincide has already been observed by Hopkins and Lurie in [HL13, Ex 4.3.11] in the context of *ambidexterity*, generalizing the notion of biproducts to (co)limits indexed by higher groupoids.

We close this section with some open questions related to the ∞-categorical model of profunctors and generalized species. Even though the model of [FGH24] is bicategorical in a full $(2, 2)$-categorical sense, our model is only ∞-categorical in the $(\infty, 1)$-categorical sense. We expect that the Lafont ∞-category $\infty \mathrm{Prof}$ can be extended to a "Lafont $(\infty, 2)$-category", but doing so was beyond the scope of this thesis, as the theory of symmetric monoidal structures is not as developed in the (∞, n)-categorical setting for $n \geq 2$ as it is for $n = 1$.

Here are some ideas of how this we could extend our result. The foundations of the theory of *enriched ∞-categories* have been laid out by Gepner and Haugseng in [GH15]. The ∞-category $\infty \mathrm{Prof}$, being is closed symmetric monoidal, can be seen as enriched over itself. Since the composite functor $\infty \mathrm{Prof} \xrightarrow{\subseteq} \infty \mathrm{Cat}_{\mathrm{cc}} \hookrightarrow \infty \mathrm{Cat}_{\mathcal{U}_1}$ is lax symmetric monoidal, this enrichment induces an enrichment over $\infty \mathrm{Cat}_{\mathcal{U}_1}$, so $\infty \mathrm{Prof}$ can be seen in this way as an $(\infty, 2)$-category. From then on we still need to check that the adjunction

$$\infty \mathrm{Prof} \underset{\underset{\perp}{\longleftarrow}}{\overset{\mathrm{Sym}_{\mathrm{cc}}}{\longrightarrow}} \mathrm{CMon}\left(\infty \mathrm{Prof}^{\otimes}\right)$$

can be extended to an $(\infty, 2)$-categorical adjunction.

Open problem 7.2.30. Show that the above defines an $(\infty, 2)$-categorical model of Linear Logic, and even of Differential Linear Logic.

Remark 7.2.31. If X and Y are small ∞-groupoids, then there is a chain of equivalences

$$
\begin{aligned}
\mathrm{Fun}_{\mathrm{cc}}\left(\mathcal{P}(X), \mathcal{P}(Y)\right) &\simeq \mathrm{Fun}\left(X, \mathcal{P}(Y)\right) \\
&\simeq \mathrm{Fun}\left(X \times Y^{\mathrm{op}}, \mathcal{S}\right) \\
&\simeq \mathrm{Fun}\left(X \times Y, \mathcal{S}\right) && (Y \simeq Y^{\mathrm{op}}) \\
&\simeq \mathcal{S}_{/X \times Y} && (\text{fact } 6.4.27)
\end{aligned}
$$

So the morphisms in $\infty \mathrm{Prof}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ correspond to spans from X to Y in $\mathcal{S}$. With more effort, it can be shown that the composition of profunctors corresponds under this equivalence to the composition of spans, so the model of theorem 7.2.26 can be seen as an ∞-categorical generalization of the wild categorical model of theorem 5.7.33 in the case that the universe $\mathcal{V}$ is the type of finite sets.

Despite the above remark, directly comparing the extensional approach of ∞-profunctors and the intensional approach of spans between ∞-groupoids is more complicated:

Open problem 7.2.32. ∞-categories of spans have been defined by Haugseng in [Hau18]. Since then, they were proven to have an $(\infty, 2)$-categorical universal property in [EH23], from which one can expect to be able to define a functor $\mathrm{Span}(\mathcal{S}) \to \infty \mathrm{Prof}$. Show that this functor is fully faithful, with essential image the ∞-categories of the form $\mathcal{P}(X)$ for X an ∞-groupoid.

Remark 7.2.33. A similar problem would be to compare our extensional definition of the ∞-category $\infty\,\mathrm{Prof}$ (as a subcategory of $\infty\,\mathrm{Cat_{cc}}$) to a more intensional, where the objects would really be small ∞-categories $\mathcal{C}, \mathcal{D}$ and the morphisms given by actual profunctors $F : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$. However, I could not find a direct definition of such an ∞-category in the literature, neither could I exhibit a satisfying definition. The closest construction I could find is the definition of $(\infty, 2)$-categories of *bimodules* in [Hau16] in the context of enriched ∞-categories. The difficulty of bridging the extensional and intensional perspectives for ∞-profunctors (or even of stating formally what this would mean) has also been noted in [BS18].

Directly following open problem 7.2.32, the authors of [EH23] also define an $(\infty, 2)$-category of polynomials (which they call *bispans*) by realizing a polynomial

$$I \xleftarrow{\;s\;} E \xrightarrow{\;p\;} B \xrightarrow{\;t\;} J$$

as a span of spans:
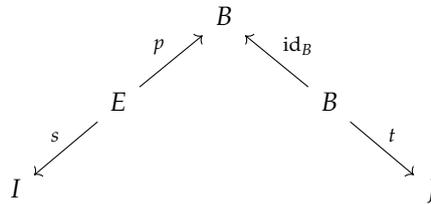
Remark 7.2.34. The fact that the composition of such spans of spans corresponds to the composition of polynomials seems to originate in [Str20].

They moreover give it an $(\infty, 2)$-categorical universal property. From then, it would also be interesting to prove the following:

Open problem 7.2.35. Show that there is a fully faithful functor between $(\infty, 2)$-categories

$$\mathrm{Poly_{fin}} \to \infty\,\mathrm{Prof}_{\mathrm{Sym_{cc}}}$$

(where $\mathrm{Poly_{fin}}$ denotes the $(\infty, 2)$-category of ∞-groupoids and *finitary* polynomials) whose essential image consists of the ∞-categories of the form $\mathcal{P}(X)$ for X an ∞-groupoid.

This would completely bridge the gap between the wild categorical story of chapter 5 and the ∞-categorical one we just presented in this section.

Going even further, recall from remark 3.11.20 that Fiore has shown in [Fio14] that generalized species between groupoids correspond to analytic functors between their presheaf categories. In [**gepner**], Gepner, Haugseng and Kock have developed a theory of analytic functors in the ∞-categorical setting, based on the theory of polynomials rather than generalized species (since the two should coincide if one is to believe in open problem 7.2.35). In particular, they show that for every ∞-groupoids X, Y, there is an equivalence of ∞-categories

$$\mathrm{AnFun}(\mathcal{S}_{/X}, \mathcal{S}_{/Y}) \simeq \mathrm{Poly_{fin}}(X, Y)$$

between the ∞-category of analytic functors from $\mathcal{S}_{/X}$ to $\mathcal{S}_{/Y}$ (and cartesian natural transformations) and the ∞-category of finitary polynomials from X to Y. They moreover show that this is equivalence is compatible with the composition of analytic functors and the composition of polynomials, but they do not show that this is true in a homotopy-coherent way. In other words the following seems still open:

Open problem 7.2.36. Enhance the above equivalence to an equivalence of $(\infty, 2)$-categories

$$\mathrm{An} \simeq \mathrm{Poly}_{\mathrm{Fin}},$$

where An denotes the $(\infty, 2)$-category of ∞-groupoids, analytic functors and cartesian natural transformations.

7.3 Higher algebra and spectra

In this final section, we study ∞-categorical variants of the models of Linear Logic based on categories of modules and vector spaces we reviewed in section 3.10. Using the theory of commutative monoids in symmetric monoidal ∞-categories, we can readily define higher commutative groups:

Definition 7.3.1. A monoid X in $\mathcal{S}^{\times}$ is said to be a *group* if for every object x in $|M|$, the functor

$$x \cdot - : |X| \to |X|$$

given by left multiplication by x in X is a homotopy equivalence. This is the case equivalently if x admits an inverse up to isomorphism, i.e. if there exists an object x^{-1} and isomorphisms

$$x \cdot x^{-1} \simeq 1 \simeq x^{-1} \cdot x,$$

where 1 denotes the unit object of X. We write $\mathrm{Grp} \subseteq \mathrm{Mon}\,(\mathcal{S}^{\times})$ for the full subcategory spanned by the groups.

Similarly, we defined *commutative groups* to be commutative monoids satisfying the same condition, and write $\mathrm{CGrp} \subseteq \mathrm{CMon}\,(\mathcal{S}^{\times})$ for the full subcategory spanned by the commutative groups.

In the realm of homotopy theory, the ∞-category of commutative groups is often replaced by larger ∞-category: the ∞-category of *spectra*, which we now review.

7.3.1 Introducing spectra

To motivate the notion of spectrum, consider a *pointed ∞-groupoid* (X, x), i.e. $x \in X_0$. The loop space $\Omega(X, x)$ can be equipped with a canonical group structure in the sense of definition 7.3.1, i.e. with a binary law that associative up to higher coherent homotopy. The homotopy group $\pi_1(X, x)$ inherits its usual group structure from this homotopical structure by passing to classes of isomorphisms in $\Omega(X, x)$. Applying Ω twice, the group $\pi_2(X, x)$ is known to be commutative by the *Eckmann-Hilton argument*: this witnesses the fact that the group $\Omega^2(X, x)$ is "more commutative" than the group $\Omega^1(X, x)$. In general, the group $\Omega^{n+1}(X, x)$ can always equipped with a "more commutative" structure than the group $\Omega^n(X, x)$. This can be made more formal using the theory of ∞-operads: there is a sequence $E_n^{\otimes}$ of ∞-operads for $n \in \mathbb{N} \cup \{\infty\}$ such that:

1. the $E_0^{\otimes}$-monoidal ∞-categories are just ∞-categories equipped with a distinguished object **1**, and the $E_0^{\otimes}$-monoids therein are the *pointed objects*, i.e. objects x equipped with a morphism $\mathbf{1} \to x$.

2. $E_1^{\otimes} \simeq \mathrm{Assoc}^{\otimes}$.

3. $E_\infty^{\otimes} \simeq \mathrm{N}\,(\mathrm{Fin}_*)$.

4. for every $m, n \leq \infty$,

$$\mathrm{Alg}_{E_m}(\mathrm{Alg}_{E_n}(\mathcal{S}^\times)^\times) \simeq \mathrm{Alg}_{E_{m+n}}(\mathcal{S}^\times).$$

5. $E_\infty^\otimes$ is the limit of the $E_n^\otimes$.

The equivalence $\mathrm{Alg}_{E_m}(\mathrm{Alg}_{E_n}(\mathcal{S}^\times)^\times) \simeq \mathrm{Alg}_{E_{m+n}}(\mathcal{S}^\times)$ can be understood concretely when restricted to $m = n = 1$ and to sets: this is the statement that monoids in the category of monoids are precisely commutative monoids. Going one step further and taking monoids in the category of commutative monoids in sets still yields the same category of commutative monoids. When working with categories, things instead stabilize after three steps:

1. E_1-algebras in Cat are the monoidal categories,

2. E_2-algebras in Cat are the *braided* monoidal categories,

3. E_k-algebras in Cat for $k \geq 3$ are the *symmetric* monoidal categories.

When with ∞-groupoids or ∞-categories, one has to go an infinite number of steps before such a *stabilization* occurs.

We can now state the precise link between E_k-structures and iterated loop spaces, known as *May's recognition theorem*:

Fact 7.3.2 ([Lur17, Thm 5.2.6.10]). *Let X be an ∞-groupoid. The data of a* groupal E_n-algebra *structure on X is equivalent to the data of a sequence of pointed ∞-groupoids $(X_0, \ldots, X_n)$ and*

- *a homotopy equivalence $X \simeq X_0$,*

- *for every $0 \leq k < n$, a pointed homotopy equivalence $X_k \simeq \Omega(X_{k+1})$,*

such that every X_k for $k \geq 1$ is connected.
When $n = \infty$, the sequence must be infinite.

Here, "groupal" means that every element has a left and right multiplicative inverse.
This motivates the following definition:

Notation 7.3.3. Write $\mathcal{S}_* := \mathcal{S}_{1/}$ for the ∞-category of *pointed ∞-groupoids*.

Definition 7.3.4 ([Lur17, Rmk 1.4.2.25]). A *spectrum* is a sequence $(X_n)_{n \in \mathbb{N}}$ of pointed ∞-groupoids together with, for every $n \in \mathbb{N}$, a pointed homotopy equivalence $X_n \simeq \Omega(X_{n+1})$. A spectrum is said to be *connective* if for every $n \geq 1$, the ∞-groupoid X_n is connected. We write Sp for the ∞-category of spectra, i.e. the limit in ∞ Cat of the sequence

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

and $\mathrm{Sp}^{\mathrm{cn}}$ for its full subcategory spanned by the connective spectra.

We can now rephrase part of fact 7.3.2 as follows:

Fact 7.3.5. *There is an equivalence of ∞-categories:*

$$\mathrm{CGrp} \simeq \mathrm{Sp}^{\mathrm{cn}}.$$

Given a spectrum $X := (X_n)_{n \in \mathbb{N}}$ and $k \in \mathbb{N}$, we have $\pi_0(X_k) \simeq \pi_2(X_{k+2})$, so $\pi_0(X_k)$ has a canonical abelian group structure. This motivates the following:

Definition 7.3.6. Let $X := (X_n)_{n\in\mathbb{N}}$ be a spectrum and $k \in \mathbb{Z}$. The k-th *homotopy group* of X is the abelian group defined as $\pi_{k+n}(X_n)$ for n big enough that $k + n \geq 2$ (note that the resulting group does not depend on the choice of such an n).

Remark 7.3.7. Since every spectrum comes with a sequence of abelian groups indexed by $\mathbb{Z}$, spectra can be thought as similar to chain complexes from homological algebra. Under this analogy, the connective spectra (and hence the commutative groups) correspond to chain complexes that are trivial in negative degree. We will mention how to make this analogy formal in remark 7.3.25.

Just like the category of abelian groups is the prototypical *abelian category*, the ∞-category of spectra is the prototypical example of a *stable ∞-category*:

Definition 7.3.8. An ∞-category $\mathcal{C}$ is said to be *pointed* if it admits an object that is both initial and terminal.

Definition 7.3.9. An ∞-category $\mathcal{C}$ with finite limits and finite colimits is said to be *stable* if:

1. it is pointed,

2. any commutative square

$$
\begin{array}{ccc}
a & \longrightarrow & b \\
\downarrow & & \downarrow \\
c & \longrightarrow & d
\end{array}
$$

in $\mathcal{C}$ is a pullback square if and only if it is a pushout square. In other words, the canonical morphism $a \to b \times_d c$ is an isomorphism if and only if the canonical morphism $b \sqcup_a c \to d$ is an isomorphism.

Fact 7.3.10 ([Lur17, Cor 1.4.2.17]). *The ∞-category* Sp *is stable.*

Remark 7.3.11. Condition 2 in definition 7.3.9 can be thought of as a more extreme variant of the existence of biproducts in abelian categories: in particular it can be shown that every stable ∞-category admits biproducts. However the condition of being stable is much stronger, and only really makes sense for ∞-categories: indeed, a stable n-category for $n < \infty$ is necessarily a poset.

7.3.2 Tensor products of commutative groups and spectra

To build higher models of Linear Logic based on commutative groups or spectra, we need suitable symmetric monoidal structures on these categories. Somewhat surprisingly, Lurie describes such a structure using the symmetric monoidal structure on $\infty\,\mathrm{Cat}_{cc}$ we described in section 7.2.1. The idea is the following: many presentable ∞-categories of interest are *idempotent objects* under the tensor product in $\mathrm{Pr}^{L\,\otimes}$, and every idempotent object can be endowed with a canonical commutative monoid structure.

Definition 7.3.12 ([Lur17, Def 4.8.2.1, Def 4.8.2.8]). Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. An *idempotent object* in $\mathcal{C}^{\otimes}$ is an object x of $\mathcal{C}^{\otimes}$ together with a morphism $e : \mathbf{1} \to x$ such that the induced morphisms

$$
x \simeq x \otimes \mathbf{1} \xrightarrow{\mathrm{id}_x \otimes e} x \otimes x \quad x \simeq \mathbf{1} \otimes x \xrightarrow{e \otimes \mathrm{id}_x} x \otimes x
$$

are isomorphisms in $\mathcal{C}$.

We say that a commutative monoid M in $\mathcal{C}^{\otimes}$ is an *idempotent commutative monoid* if its multiplication morphism $|M| \otimes |M| \to |M|$ is an isomorphism in $\mathcal{C}$, and write $\mathrm{CMon}^{\mathrm{idem}}(\mathcal{C}^{\otimes}) \subseteq \mathrm{CMon}(\mathcal{C}^{\otimes})$ for the full subcategory spanned by the idempotent commutative monoids.

The canonical commutative monoid structure on idempotent objects is given by the following:

Fact 7.3.13 ([Lur17, Def 4.8.2.9]). *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. The composite functor*

$$\mathrm{CMon}^{\mathrm{idem}}\left(\mathcal{C}^{\otimes}\right) \subseteq \mathrm{CMon}\left(\mathcal{C}^{\otimes}\right) \to \mathcal{C}_{\mathbf{1}/},$$

(where the last step is given by only remembering the unit $e : \mathbf{1} \to |M|$ of a commutative monoid M) is fully faithful, and its essential image consists of the idempotent objects in $\mathcal{C}$.

Fact 7.3.14. *Let $\mathcal{C}$ be a presentable ∞-category equipped with an object $\mathbf{1}$. The object $\mathbf{1}$ determines a unique cocontinuous functor $F : \mathcal{S} \to \mathcal{C}$ mapping 1 to $\mathbf{1}$, since $\mathcal{S} \simeq \mathcal{P}(1)$. If $(\mathcal{C}, F)$ is an idempotent object of Pr^{L}, then $\mathcal{C}$ admits a closed symmetric monoidal structure whose unit is the object $\mathbf{1}$, and for which the functor $F : \mathcal{S} \to \mathcal{C}$ extends to a symmetric monoidal $F : \mathcal{S}^{\times} \to \mathcal{C}^{\otimes}$.*

Proof. By fact 7.3.13, $(\mathcal{C}, \mathbf{1})$ can be promoted to a commutative monoid in $\mathrm{Pr}^{\mathrm{L}\,\otimes}$. Unfolding definitions, such a commutative monoid corresponds precisely to a commutative monoid in $\infty\,\mathrm{Cat}$ whose underlying ∞-category is presentable and for which the tensor product functor is bicontinuous, i.e. a presentably symmetric monoidal ∞-category, which in particular is closed. Moreover, the functor $F : \mathcal{S} \to \mathcal{C}$ is by definition a morphism in $\mathrm{Pr}^{\mathrm{L}}_{/\mathcal{S}}$, so it also promotes to a morphism of commutative monoids in Pr^{L}, hence a symmetric monoidal functor. $\square$

Corollary 7.3.15. *Every idempotent object $\mathcal{C}$ in Pr^{L} induces a linear/non-linear adjunction*

$$\mathcal{S}^{\times} \underset{\xleftarrow{\hspace{1.5em}}}{\overset{F}{\underset{\perp}{\xrightarrow{\hspace{1.5em}}}}} \mathcal{C}^{\otimes}$$

Proof. This is fact 7.3.14, combined with the adjoint functor theorem to get the right adjoint by virtue of F preserving colimits (fact 6.8.15). $\square$

Example 7.3.16 ([Lur17, Rmk 4.8.2.14]). The ∞-category $\mathcal{S}_{*}$ is presentable, and an idempotent object of Pr^{L} with unit the 0-*sphere* S^{0}, i.e. the discrete ∞-groupoid on two points. Indeed, for every presentable ∞-category $\mathcal{C}$, $\mathcal{S}_{*} \otimes \mathcal{C}$ is the universal pointed ∞-category on $\mathcal{C}$, so $\mathcal{S}_{*} \otimes \mathcal{C} \simeq \mathcal{C}$ whenever $\mathcal{C}$ is already pointed, and in particular $\mathcal{S}_{*} \otimes \mathcal{S}_{*} \simeq \mathcal{S}_{*}$.

Example 7.3.17 ([Lur17, Prop 4.8.2.18]). The ∞-category Sp is presentable, and an idempotent object of Pr^{L} whose unit is given by the *sphere spectrum* (the free commutative group on a point). This is because for every presentable ∞-category $\mathcal{C}$, $\mathrm{Sp} \otimes \mathcal{C}$ is the universal stable ∞-category on $\mathcal{C}$, so $\mathrm{Sp} \otimes \mathcal{C} \simeq \mathcal{C}$ whenever is already stable, and in particular $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$.

Remark 7.3.18. Interestingly, the same thing can be said about the category of abelian groups: Ab is an idempotent object in the $(2, 1)$-category of presentable 1-categories, and the usual tensor product of abelian groups can thus be characterized as the unique bicontinuous functor $\mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ for which $\mathbb{Z}$ is a unit.

Fact 7.3.19 ([Lur17, Cor 7.1.4.13, Lem 7.1.1.7]). *The ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ is presentable, and the symmetric monoidal structure on Sp restricts to $\mathrm{Sp}^{\mathrm{cn}}$.*

Proposition 7.3.20. *The ∞-category $\mathrm{CMon}\left(\mathcal{S}^{\times}\right)$ is an idempotent object of Pr^{L}.*

Proof. Informally, we have

$$
\begin{aligned}
\mathrm{CMon}\left(\mathcal{S}^{\times}\right) \otimes \mathrm{CMon}\left(\mathcal{S}^{\times}\right) &\simeq \mathrm{Alg}_{E_{\infty}}\left(\mathcal{S}^{\times}\right) \otimes \mathrm{Alg}_{E_{\infty}}\left(\mathcal{S}^{\times}\right) \\
&\simeq \mathrm{Alg}_{E_{\infty}}\left(\mathrm{Alg}_{E_{\infty}}\left(\mathcal{S}^{\times}\right)^{\times}\right) \\
&\simeq \mathrm{Alg}_{E_{\infty+\infty}}\left(\mathcal{S}^{\times}\right) \\
&\simeq \mathrm{Alg}_{E_{\infty}}\left(\mathcal{S}^{\times}\right) \\
&\simeq \mathrm{CMon}\left(\mathcal{S}^{\times}\right)
\end{aligned}
$$

$\square$

Remark 7.3.21. The symmetric monoidal structures on $\mathrm{CMon}\,(\mathcal{S}^{\times})$ and CGrp could also be recovered using the characterization of these ∞-categories as ∞-categories of models for *higher Lawvere theories* [Ber20]. From this point of view, the proof of proposition 7.3.20 seems to be an ∞-categorical variant a standard reasoning giving closed symmetric monoidal structures on categories of models of *commutative algebraic theories* [nLa25a; Kei78].

Combining all the above examples, we have:

Theorem 7.3.22. *There is a chain of symmetric monoidal left adjoint functors:*

$$\mathcal{S}^{\times} \underset{\bot}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{CMon}\,(\mathcal{S})^{\otimes} \underset{\bot}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{CGrp}^{\otimes} \quad \simeq \quad \mathrm{Sp}^{\mathrm{cn}\,\otimes} \underset{\bot}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Sp}^{\otimes}$$

thus inducing linear/non-linear adjunctions between $\mathcal{S}$ and each of the ∞-categories $\mathrm{CMon}\,(\mathcal{S})$, CGrp *and* Sp.

These linear/non-linear adjunctions can be thought of as ∞-categorical variants of the model of proposition 3.10.6. Going further, we can define "higher commutative rings" and their "higher modules" as commutative monoids in $\mathrm{Sp}^{\otimes}$ and their left modules in $\mathrm{Sp}^{\otimes}$. In the homotopical literature, these are more often referred to, respectively, as E_{∞}-ring spectra and E_{∞}-module spectra.

Fact 7.3.23 ([Lur17, Thm 4.5.3.1]). *Given an E_{∞}-ring spectrum R, and writing Mod_R for the ∞-category of left modules on R in Sp, Mod_R admits a canonical symmetric monoidal structure, and there is a symmetric monoidal left adjoint to the forgetful functor $\mathrm{Mod}_R^{\otimes} \to \mathrm{Sp}^{\otimes}$.*

The following can really be thought as the proper ∞-categorical version of proposition 3.10.6.

Theorem 7.3.24. *For every E_{∞}-ring spectrum R, there is a linear/non-linear adjunction:*

$$\mathcal{S}^{\times} \underset{\bot}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Mod}_R^{\otimes}$$

Proof. This is obtained by composing the symmetric monoidal adjunction of fact 7.3.23 with the linear/non-linear adjunction between $\mathcal{S}$ and Sp of theorem 7.3.22. $\square$

Remark 7.3.25. The analogy between spectra and chain complexes of remark 7.3.7 can be made more formal using the theory if module spectra. Given a (usual, set-theoretic) ring R, one can see R as a connective discrete E_{∞}-ring spectrum. It can then be shown that the ∞-category Mod_R is equivalent to the homotopy coherent nerve of a simplicial version of the category of unbounded chain complexes of R-modules (in the usual sense). In other words, Mod_R is the localization of the category of chain complexes of R-modules at the class of quasi-isomorphisms (see remark 6.4.25).

Since all the above are presentably symmetric monoidal ∞-categories, theorem 7.1.15 also gets us the following ∞-categorical version of proposition 3.10.8:

Theorem 7.3.26. *The ∞-categories $\mathcal{S}_*$, $\mathrm{CMon}\,(\mathcal{S}^{\times})$, CGrp, Sp and Mod_R for R an E_{∞}-ring spectrum all admit cofree commutative comonoids, and are thus Lafont ∞-categories.*

We close this thesis with a few remarks about spectra and the models we just described. First we recall an important construction in the homotopy theory of ∞-categories:

Definition 7.3.27. Let $\mathcal{C}$ be a pointed ∞-category with finite limits and finite colimits, and X an object of $\mathcal{C}$. We define the *suspension* of X and the *loop space* of X respectively as:

$$\Sigma X := 1 \sqcup_X 1$$
$$\Omega X := 1 \times_X 1$$

Fact 7.3.28. *The suspension and loop space define adjoint functors:*

$$\mathcal{C} \xrightarrow[\underset{\Omega}{\longleftarrow}]{\overset{\Sigma}{\longrightarrow}} {\scriptstyle \perp} \; \mathcal{C}$$

because both $\mathrm{Hom}_{\mathcal{C}}(x, 1 \times_y 1)$ *and* $\mathrm{Hom}_{\mathcal{C}}(1 \sqcup_x 1, y)$ *are homotopy equivalent to the data of a homotopy commutative square*

$$
\begin{array}{ccc}
x & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & y
\end{array}
$$

By definition, in a stable ∞-category the above adjunction must be an equivalence. As it turns out, this is also a characterization of stable ∞-categories:

Fact 7.3.29. *A pointed ∞-category with finite limits and colimits is stable if and only if the adjunction $\Sigma \dashv \Omega$ is an equivalence.*

Notation 7.3.30. The functors involved in the adjunction between $\mathcal{S}_*$ and Sp are called:

- the *infinite suspension* functor, denoted $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$,

- the *infinite loop space* functor, denoted $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$.

Remark 7.3.31. If X is a pointed *connected* ∞-groupoid, i.e. satisfying $\pi_0(X) = 1$, then the free commutative monoid on X is already a group by the James construction [DH21]. This fact can be used to give a formula for the comonad $\Sigma^\infty \Omega^\infty \mathbb{X}$ when $\mathbb{X}$ is a connected *suspension spectrum* (i.e. lying in the image of Σ^∞): under that assumption, we have

$$\Sigma^\infty \Omega^\infty \mathbb{X} \simeq \bigoplus_{n \geq 1} \mathbb{X}^{\otimes n} /\!\!/ \mathfrak{S}_n$$

by [AC20, Ex 1.2.6], a formula very reminiscent of the one for the free commutative monoid (with specified unit) on $\mathbb{X}$ (note that taking free commutative monoid is normally a monad, not a comonad).

For that reason, the comonad $\Sigma^\infty \Omega^\infty$ has been conjectured to be an exponential comonad for a homotopical model of Linear Logic in the context of Goodwillie's calculus of functors [nLa25b]. However, the symmetric monoidal structure on $\mathcal{S}_*$ is not cartesian, so its adjunction with Sp is not a linear/non-linear adjunction. Instead, taking the comonad $\Sigma^\infty \Omega^\infty (-)_+$ induced by the adjunction between $\mathcal{S}$ and Sp gives the formula

$$\Sigma^\infty \Omega^\infty \mathbb{X}_+ \simeq \Sigma^\infty \Omega^\infty \mathbb{X} \oplus \mathbb{S} \simeq \bigoplus_{n \geq 0} \mathbb{X}^{\otimes n} /\!\!/ \mathfrak{S}_n$$

for $\mathbb{X}$ a connected suspension spectrum. The added term $n = 0$ makes the right-hand side equivalent to the free commutative monoid on $\mathbb{X}$.

This phenomenon is truly exclusive to the homotopical setting: if one wanted to obtain a similar result for abelian groups for instance, the result is still true but the assumption of $\mathbb{X}$ being connected is only satisfied by the trivial group. For general $\mathbb{X}$ there is no reason to expect the "free commutative group on a commutative group" comonad and the "free commutative algebra" monad to coincide.

Remark 7.3.32. The ∞-category Sp is not ∗-autonomous, as it is presentable (see the proof of proposition 3.10.9). However, it is known in stable homotopy theory that every *finite* spectrum is dualizable (with a finite dual), so the ∞-category of finite spectra is compact closed, so it is ∗-autonomous with the unit spectrum $\mathbb{S}$ as a dualizing object. Moreover using the adjoint equivalence

$$\Sigma : \mathrm{Sp}_{\mathrm{Fin}} \rightleftarrows \mathrm{Sp}_{\mathrm{Fin}} : \Omega,$$

we also get that every $\Sigma^n \mathbb{S}$, with $n \in \mathbb{Z}$, is a dualizing object in $\mathrm{Sp}_{\mathrm{Fin}}$, thus giving an example of a ∗-autonomous ∞-category with multiple non-isomorphic dualizing objects.

Remark 7.3.33. Beyond the linear/non-linear adjunctions we described above, the chains of adjunctions of theorem 7.3.22 and corollary 7.2.11 and give multiple examples of resource modalities in the sense of [MT10]. In particular, the chain of adjunctions

$$\mathcal{S} \underset{}{\overset{(-)_+}{\rightleftarrows}} \mathcal{S}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Sp}$$

decomposes the linear exponential comonad $\Sigma^\infty \Omega^\infty (-)_+$ into an *affine* exponential $(-)_+$ on $\mathcal{S}_*$ followed by a *relevant* exponential $\Sigma^\infty \Omega^\infty$ on Sp. From the point of view of Linear Logic, this corresponds to considering separately the dynamics of weakening and contraction, as first studied in [Jac94].

Chapter 8

Perspectives

We close this thesis with a few perspectives of possible future developments.

8.1 Free (co)monoids in homotopy type theory

Even though it is seemingly impossible to define fully homotopy-coherent structures in homotopy type theory, it is still possible to do it for low-dimensional structures. For instance, ordinary categories and bicategories can be defined in a fully homotopy-coherent way in homotopy type theory, because they are of finite homotopy level. In particular, given a 1-type X, it should be possible to state that the 1-type

$$\mathrm{HMul}\,(X) := \sum_{E:\mathrm{Fin}} (E \to X)$$

is the free commutative monoid on X in the bicategory of 1-types. However, this seemingly mundane result does not appear in the literature on homotopy type theory. Having tried to give a direct proof by hand in the proof assistant Cubical Agda, it seems it may not be so straightforward to prove after all, even in the case where X is the unit type, and I think a complete proof may require some interesting ideas.

Open problem 8.1.1. Show that n-type $\mathrm{HMul}\,(X)$ is the free symmetric monoidal n-type on X, for low values of n.

8.2 Higher differential linear logic

We already mentioned in open problem 7.2.30 that it would be interesting to generalize the results of chapter 7 to the setting of $(\infty, 2)$-categories to be able to define ∞-categorical models of *differential* linear logic. We'd like to also mention the recent work of Kerjean, Maestracci and Rogers on "Functorial Models of Differential Linear Logic" [KMR25], where they give another abstract categorical axiomatization for models of differential linear logic — building upon the structure of linear/non-linear adjunctions — which could lead to another way of approaching ∞-categorical models of differential linear logic.

8.3 (∞,2)-categorical models and Goodwillie's calculus

We opened this thesis with the very idea that Goodwillie's calculus of functors could fit into an ∞-categorical model of linear logic. With all the background reviewed in chapters 6 and 7, we are now ready to say a bit more about this.

In Goodwillie's calculus, a functor $f : \mathcal{C} \to \mathcal{D}$ is considered to be "linear" if it maps pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$. Such functors are said to be 1-*excisive*. This seemingly bizarre condition becomes way more natural when $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-categories: in that case, a commutative square is a pushout square if and only if it is a pullback square, so the 1-excisive functors are precisely the functors that preserve finite limits, or equivalently in that case finite colimits. In particular, the functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is 1-excisive if and only if the ∞-category $\mathcal{C}$ is stable. From the point of view of homotopy theory, stable ∞-categories are much easier to understand than non-stable ones, so the idea of approximating arbitrary functors by 1-excisive ones makes a lot of sense: it is akin to approximating a smooth function by its differential. More generally, Goodwillie's calculus can informally be summarized using the following analogy table, taken from [Lur17, Chapter 6]:

Differential calculus	Goodwillie's calculus of functors
Smooth manifold M	(Finitely) presentable ∞-category $\mathcal{C}$
Smooth function $f : M \to N$	Functor $F : \mathcal{C} \to \mathcal{D}$ preserving filtered colimits
Point $x \in M$	Object $x \in \mathcal{C}_0$
Real vector space	Stable ∞-category
Real number line $\mathbb{R}$	Stable ∞-category of spectra Sp
Linear map of vector spaces	Finite (co)limit preserving functor between stable ∞-categories

Table 8.1: Analogy between differential calculus and Goodwillie's calculus of functors.

The hope is the following:

Open problem 8.3.1. Can one construct a linear/non-linear adjunction in which the linear category consists of stable ∞-categories and 1-excisive functors, and the non-linear category consists of finitely presentable ∞-categories and functors that preserve filtered colimits?

Going even further, the notion of 1-excisive functor can be generalized to n-excisive functors, which can be thought of as "polynomials of degree n". From any functor $F : \mathcal{C} \to \mathcal{D}$, one can build a tower of natural transformations

$$F \longrightarrow \cdots \qquad \cdots \longrightarrow P_2(F) \longrightarrow P_1(F) \longrightarrow P_0(F)$$

where $P_n(F)$ is the "best n-excisive approximation of F", this is in a sense a *Taylor tower* for the functor F. In view of the bicategorical structures of differential linear logic highlighted in [FGH24], a natural subsequent question would be:

Open problem 8.3.2. Is the Taylor tower of the functor F above an instance of theory of Taylor expansions in models of differential linear logic?

In some very recent work [BBC23], Bauer, Burke and Ching have described how Goodwillie's calculus is an instance of a *tangent ∞-category*, an ∞-categorical generalization of the *tangent category* structure which has also been recently investigated in connection with linear logic [CLL20]. We hope the present work can pave the way to further connect such ideas in the future.

Bibliography

[AC20]   Gregory Arone and Michael Ching. "Goodwillie calculus". In: (Jan. 23, 2020). Ed. by Haynes Miller, pp. 1–38. DOI: 10.1201/9781351251624-1 (cit. on p. 270).

[Adá94]   Jiří Adámek. *Locally presentable and accessible categories*. In collab. with Jiří Rosický. London Mathematical Society lecture note series 189. Cambridge: Cambridge University Press, 1994. 316 pp. ISBN: 978-0-521-42261-1 978-1-107-36909-2 (cit. on pp. 80, 223–225).

[AFS21]   Antoine Allioux, Eric Finster, and Matthieu Sozeau. *Types are Internal ∞-Groupoids*. Apr. 30, 2021. arXiv: 2105.00024[cs]. (Visited on 03/06/2025) (cit. on p. 123).

[AL18]   Mathieu Anel and Damien Lejay. *Exponentiable Higher Toposes*. Feb. 28, 2018. DOI: 10.48550/arXiv.1802.10425 (cit. on pp. 258, 262).

[All23]   Antoine Allioux. "Higher Structures in Homotopy Type Theory". PhD thesis. Université Paris Cité, July 2023. URL: https://theses.hal.science/tel-04506555 (cit. on p. 123).

[ARV10]   J. Adamek, J. Rosicky, and E. M. Vitale. "What are sifted colimits?" In: *Theory and Applications of Categories* [*electronic only*] 23 (2010). Publisher: Mount Allison University, Department of Mathematics and Computer Science, Sackville, pp. 251–260. ISSN: 1201-561X. URL: https://eudml.org/doc/233342 (visited on 09/02/2025) (cit. on p. 87).

[Bar74]   Michael Barr. "Coalgebras over a commutative ring". In: *Journal of Algebra* 32.3 (Dec. 1974), pp. 600–610. ISSN: 00218693. DOI: 10.1016/0021-8693(74)90161-6 (cit. on p. 80).

[BBC23]   Kristine Bauer, Matthew Burke, and Michael Ching. *Tangent infinity-categories and Goodwillie calculus*. Feb. 25, 2023. DOI: 10.48550/arXiv.2101.07819 (cit. on p. 274).

[BCS06]   R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. "Differential categories". In: *Mathematical Structures in Computer Science* 16.6 (Dec. 2006), p. 1049. ISSN: 0960-1295, 1469-8072. DOI: 10.1017/S0960129506005676. (Visited on 04/22/2024) (cit. on p. 89).

[Ben+92]   Nick Benton et al. "Linear λ-calculus and categorical models revisited". In: *Computer Science Logic*. Vol. 702. Series Title: Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 61–84. ISBN: 978-3-540-56992-3 978-3-540-47890-4. DOI: 10.1007/3-540-56992-8_6. URL: http://link.springer.com/10.1007/3-540-56992-8_6 (visited on 08/22/2025) (cit. on p. 41).

[Ben94]   Nick Benton. "A mixed linear and non-linear logic: Proofs, terms and models". In: *International Workshop on Computer Science Logic*. Springer. 1994, pp. 121–135 (cit. on pp. 42, 43).

[Ber20]   John D. Berman. "Higher Lawvere theories". In: *Journal of Pure and Applied Algebra* 224.9 (Sept. 1, 2020), p. 106362. ISSN: 0022-4049. DOI: `10.1016/j.jpaa.2020.106362` (cit. on p. 269).

[Bie93]   G M Bierman. "On Intuitionistic Linear Logic". PhD thesis. Dec. 1993 (cit. on pp. 32, 40, 41).

[Bie95]   G. M. Bierman. "What is a categorical model of Intuitionistic Linear Logic?" In: *Typed Lambda Calculi and Applications*. Ed. by Mariangiola Dezani-Ciancaglini and Gordon Plotkin. Berlin, Heidelberg: Springer, 1995, pp. 78–93. ISBN: 978-3-540-49178-1. DOI: `10.1007/BFb0014046` (cit. on p. 40).

[BK12]   C. Barwick and D. M. Kan. "Relative categories: Another model for the homotopy theory of homotopy theories". In: *Indagationes Mathematicae* 23.1 (Mar. 1, 2012), pp. 42–68. ISSN: 0019-3577. DOI: `10.1016/j.indag.2011.10.002`. URL: `https://www.sciencedirect.com/science/article/pii/S0019357711000620` (visited on 08/09/2025) (cit. on p. 21).

[BLO16]   Richard Blute, Rory B B Lucyshyn-Wright, and Keith O'Neill. "Derivations in codifferential categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* (2016) (cit. on p. 89).

[Blu+20]   R. F. Blute et al. "Differential Categories Revisited". In: *Applied Categorical Structures* 28.2 (Apr. 1, 2020), pp. 171–235. ISSN: 1572-9095. DOI: `10.1007/s10485-019-09572-y` (cit. on pp. 89, 90).

[Bor94a]   Francis Borceux. *Handbook of categorical algebra, Volume 1 : Basic category theory*. Encyclopedia of mathematics and its applications v. 50-51, 53 [i.e. 52]. Cambridge [England] ; New York: Cambridge University Press, 1994. DOI: `https://doi.org/10.1017/CBO9780511525858` (cit. on p. 49).

[Bor94b]   Francis Borceux. *Handbook of categorical algebra, Volume 2 : Categories and structures*. Encyclopedia of mathematics and its applications v. 50-51, 53 [i.e. 52]. Cambridge [England] ; New York: Cambridge University Press, 1994. 3 pp. DOI: `https://doi.org/10.1017/CBO9780511525865` (cit. on pp. 48, 49, 51).

[BS18]   Clark Barwick and Jay Shah. "Fibrations in ∞-Category Theory". In: *2016 MATRIX Annals*. Ed. by Jan de Gier, Cheryl E. Praeger, and Terence Tao. Cham: Springer International Publishing, 2018, pp. 17–42. ISBN: 978-3-319-72299-3. DOI: `10.1007/978-3-319-72299-3_2` (cit. on p. 264).

[Buc19]   Ulrik Buchholtz. "Higher Structures in Homotopy Type Theory". In: *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*. Springer International Publishing, 2019, pp. 151–172. ISBN: 978-3-030-15655-8. DOI: `10.1007/978-3-030-15655-8_7` (cit. on pp. 92, 123).

[Bur71]   Albert Burroni. "*T*-catégories (catégories dans un triple)". fre. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 12.3 (1971), pp. 215–321. URL: `http://eudml.org/doc/91097` (cit. on p. 69).

[But98]   Carsten Butz. "Regular Categories and Regular Logic". In: 1998. URL: `https://www.semanticscholar.org/paper/Regular-Categories-and-Regular-Logic-Butz/1eb2126e15a88180340a617001a562ac9fbc7017` (cit. on p. 52).

[CF23]   Pierre Clairambault and Simon Forest. *The Cartesian Closed Bicategory of Thin Spans of Groupoids*. Jan. 27, 2023. DOI: `10.48550/arXiv.2301.11860` (cit. on pp. 5, 70).

[CG24]   Matteo Capucci and Bruno Gavranović. *Actegories for the Working Amthematician*. Sept. 10, 2024. DOI: 10.48550/arXiv.2203.16351 (cit. on p. 244).

[Cis19]   Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. 1st ed. Cambridge University Press, Apr. 30, 2019. ISBN: 978-1-108-58873-7 978-1-108-47320-0. DOI: 10.1017/9781108588737. URL: https://www.cambridge.org/core/product/identifier/9781108588737/type/book (cit. on p. 217).

[CK18]   Paolo Capriotti and Nicolai Kraus. "Univalent higher categories via complete Semi-Segal types". In: *Proceedings of the ACM on Programming Languages* 2 (POPL Jan. 2018), pp. 1–29. ISSN: 2475-1421. DOI: 10.1145/3158132 (cit. on pp. 5, 7, 123–125).

[CLL20]   Robin Cockett, Jean-Simon Pacaud Lemay, and Rory B. B. Lucyshyn-Wright. "Tangent Categories from the Coalgebras of Differential Categories". In: *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*. Ed. by Maribel Fernández and Anca Muscholl. Vol. 152. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020, 17:1–17:17. ISBN: 978-3-95977-132-0. DOI: 10.4230/LIPIcs.CSL.2020.17 (cit. on p. 274).

[CLW93]   Aurelio Carboni, Stephen Lack, and R. F. C. Walters. "Introduction to extensive and distributive categories". In: *Journal of Pure and Applied Algebra* 84.2 (Feb. 3, 1993), pp. 145–158. ISSN: 0022-4049. DOI: 10.1016/0022-4049(93)90035-R. URL: https://www.sciencedirect.com/science/article/pii/002240499390035R (visited on 05/26/2025) (cit. on p. 67).

[CM20]   James Clift and Daniel Murfet. "Cofree coalgebras and differential linear logic". In: *Mathematical Structures in Computer Science* 30.4 (Apr. 2020), pp. 416–457. ISSN: 0960-1295, 1469-8072. DOI: 10.1017/S0960129520000134. URL: https://www.cambridge.org/core/journals/mathematical-structures-in-computer-science/article/abs/cofree-coalgebras-and-differential-linear-logic/FDFA007DF709AE4600E43C08B25479EB (visited on 08/26/2025) (cit. on p. 89).

[COP24]   Pierre Clairambault, Federico Olimpieri, and Hugo Paquet. *From Thin Concurrent Games to Generalized Species of Structures (Extended Version)*. July 19, 2024. DOI: 10.48550/arXiv.2302.01082 (cit. on p. 5).

[CW05a]   Mario Caccamo and Glynn Winskel. "Limit Preservation from Naturality". In: *Electronic Notes in Theoretical Computer Science*. Proceedings of the 10th Conference on Category Theory in Computer Science (CTCS 2004) 122 (Mar. 7, 2005), pp. 3–22. ISSN: 1571-0661. DOI: 10.1016/j.entcs.2004.06.048 (cit. on p. 221).

[CW05b]   Gian Cattani and Glynn Winskel. "Profunctors, Open Maps and Bisimulation". In: *Mathematical Structures in Computer Science* 15 (June 1, 2005), pp. 553–614. DOI: 10.1017/S0960129505004718 (cit. on p. 86).

[DH21]   Sanath Devalapurkar and Peter Haine. "On the James and Hilton-Milnor splittings, and the metastable EHP sequence". In: *Documenta Mathematica* 26 (2021), pp. 1423–1464. ISSN: 1431-0635, 1431-0643. DOI: 10.4171/dm/845 (cit. on p. 270).

[EH23]   Elden Elmanto and Rune Haugseng. "On distributivity in higher algebra I: the universal property of bispans". In: *Compositio Mathematica* 159.11 (Nov. 2023), pp. 2326–2415. ISSN: 0010-437X, 1570-5846. DOI: 10.1112/S0010437X23007388 (cit. on pp. 263, 264).

[Ehr02]   Thomas Ehrhard. "On Köthe sequence spaces and linear logic". In: *Mathematical Structures in Computer Science* 12.5 (Oct. 2002), pp. 579–623. ISSN: 1469-8072, 0960-1295. DOI: 10.1017/S0960129502003729 (cit. on p. 81).

[Ehr05]   Thomas Ehrhard. "Finiteness spaces". In: *Mathematical Structures in Computer Science* 15.4 (Aug. 2005), pp. 615–646. ISSN: 0960-1295, 1469-8072. DOI: 10.1017/S0960129504004645 (cit. on p. 89).

[Ehr12]   Thomas Ehrhard. "The Scott model of linear logic is the extensional collapse of its relational model". In: *Theoretical Computer Science* 424 (Mar. 23, 2012), pp. 20–45. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2011.11.027. URL: https://www.sciencedirect.com/science/article/pii/S0304397511009467 (visited on 05/28/2025) (cit. on p. 77).

[Ehr18]   Thomas Ehrhard. "An introduction to differential linear logic: proof-nets, models and antiderivatives". In: *Mathematical Structures in Computer Science* 28.7 (Aug. 2018), pp. 995–1060. ISSN: 0960-1295, 1469-8072. DOI: 10.1017/S0960129516000372 (cit. on p. 89).

[EM45]   Samuel Eilenberg and Saunders MacLane. "General theory of natural equivalences". In: *Transactions of the American Mathematical Society* 58.0 (1945), pp. 231–294. ISSN: 0002-9947, 1088-6850. DOI: 10.1090/S0002-9947-1945-0013131-6. URL: https://www.ams.org/tran/1945-058-00/S0002-9947-1945-0013131-6/ (visited on 08/07/2025) (cit. on p. 13).

[ER03]   Thomas Ehrhard and Laurent Regnier. "The differential lambda-calculus". In: *Theoretical Computer Science* 309.1 (Dec. 2, 2003), pp. 1–41. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(03)00392-X (cit. on p. 89).

[ER06]   T. Ehrhard and L. Regnier. "Differential interaction nets". In: *Theoretical Computer Science*. Logic, Language, Information and Computation 364.2 (Nov. 6, 2006), pp. 166–195. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2006.08.003 (cit. on p. 89).

[FGH24]   M. Fiore, N. Gambino, and M. Hyland. *Monoidal bicategories, differential linear logic, and analytic functors*. 2024. arXiv: 2405.05774 [math.CT]. URL: https://arxiv.org/abs/2405.05774 (cit. on pp. 4, 6, 7, 82, 85, 90, 91, 174, 175, 177–179, 184, 186, 187, 262, 263, 273).

[FGP23]   Marcelo Fiore, Zeinab Galal, and Hugo Paquet. *Stabilized profunctors and stable species of structures*. version: 2. Oct. 9, 2023. DOI: 10.48550/arXiv.2303.04795. (Visited on 01/05/2024) (cit. on pp. 7, 71, 82).

[Fin+21]   Eric Finster et al. "A Cartesian Bicategory of Polynomial Functors in Homotopy Type Theory". In: *Electronic Proceedings in Theoretical Computer Science* 351 (Dec. 29, 2021), pp. 67–83. ISSN: 2075-2180. DOI: 10.4204/EPTCS.351.5. URL: http://arxiv.org/abs/2112.14050 (cit. on pp. 1, 4, 5, 7, 124, 168, 169).

[Fio+07]   M. Fiore et al. "The cartesian closed bicategory of generalised species of structures". In: *Journal of the London Mathematical Society* 77.1 (Oct. 20, 2007), pp. 203–220. ISSN: 00246107. DOI: 10.1112/jlms/jdm096. (Visited on 06/29/2023) (cit. on pp. 4, 82, 84–86, 172).

[Fio+17]   Marcelo Fiore et al. "Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures". In: *Selecta Mathematica* 24.3 (Nov. 20, 2017), pp. 2791–2830. ISSN: 1022-1824, 1420-9020. DOI: 10.1007/s00029-017-0361-3 (cit. on pp. 82, 85).

[Fio14]   Marcelo Fiore. "Analytic functors between presheaf categories over groupoids". In: *Theoretical Computer Science*. Models of Interaction: Essays in Honour of Glynn Winskel 546 (Aug. 21, 2014), pp. 120–131. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2014.03.004 (cit. on pp. 82, 86, 174, 264).

[Fre70]   Peter Freyd. "Homotopy is not concrete". In: *The Steenrod Algebra and Its Applications: A Conference to Celebrate NE Steenrod's Sixtieth Birthday: Proceedings of the Conference held at the Battelle Memorial Institute, Columbus, Ohio March 30th–April 4th, 1970*. Springer. 1970, pp. 25–34. DOI: 10.1007/BFb0058516 (cit. on p. 165).

[Gal21]   Zeinab Galal. "Bicategorical orthogonality constructions for linear logic". These de doctorat. Université Paris Cité, Sept. 27, 2021. URL: https://www.theses.fr/2021UNIP7118 (visited on 01/15/2024) (cit. on pp. 5, 6, 86, 88).

[GH15]   David Gepner and Rune Haugseng. "Enriched ∞-categories via non-symmetric ∞-operads". In: *Advances in Mathematics* 279 (July 2015), pp. 575–716. ISSN: 00018708. DOI: 10.1016/j.aim.2015.02.007 (cit. on pp. 21, 263).

[GHK22]   David Gepner, Rune Haugseng, and Joachim Kock. "∞-Operads as Analytic Monads". In: *International Mathematics Research Notices* 2022.16 (Aug. 6, 2022), pp. 12516–12624. ISSN: 1073-7928, 1687-0247. DOI: 10.1093/imrn/rnaa332 (cit. on p. 175).

[GHN17]   David Gepner, R. Haugseng, and T. Nikolaus. "Lax colimits and free fibrations in ∞-categories". In: *Documenta Mathematica* (2017). URL: https://www.semanticscholar.org/paper/Lax-colimits-and-free-fibrations-in-%E2%88%9E-categories-Gepner-Haugseng/c59ed54deb97388fdb84df7c032a6d842694e84c (cit. on p. 208).

[Gir87]   Jean-Yves Girard. "Linear logic". In: *Theoretical Computer Science* 50.1 (Jan. 1, 1987), pp. 1–101. ISSN: 0304-3975. DOI: 10.1016/0304-3975(87)90045-4. URL: https://www.sciencedirect.com/science/article/pii/0304397587900454 (visited on 08/19/2025) (cit. on pp. 2, 29, 31).

[Gir88]   Jean-Yves Girard. "Normal functors, power series and λ-calculus". In: *Annals of Pure and Applied Logic* 37.2 (Feb. 1, 1988), pp. 129–177. ISSN: 0168-0072. DOI: 10.1016/0168-0072(88)90025-5 (cit. on pp. 3, 29, 86, 172).

[GJ09]   Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Reprint of the 1999 ed. Modern Birkhäuser classics. Basel Boston, Mass. Berlin: Birkhäuser, 2009. 510 pp. ISBN: 978-3-0346-0188-7 (cit. on p. 216).

[GK13]   Nicola Gambino and Joachim Kock. "Polynomial functors and polynomial monads". In: 154.1 (2013), pp. 153–192. DOI: 10.1017/S0305004112000394 (cit. on pp. 55, 166).

[GLT89]   Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science 7. Cambridge University Press, 1989 (cit. on p. 29).

[GU71]   Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*. Vol. 221. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1971. ISBN: 978-3-540-05578-5 978-3-540-36886-1. DOI: 10.1007/BFb0059396 (cit. on p. 223).

[Har18]   Yonatan Harpaz. *Answer to "How can I functorially dualise in a symmetric monoidal $(\infty, 1)$-category with duals?"* MathOverflow. July 20, 2018. URL: https://mathoverflow.net/a/306491/508115 (visited on 10/02/2025) (cit. on p. 241).

[Hat02]   Allen Hatcher. *Algebraic topology*. Cambridge ; New York: Cambridge University Press, 2002. 544 pp. ISBN: 978-0-521-79160-1 978-0-521-79540-1 (cit. on p. 15).

[Hau+23]   Rune Haugseng et al. "Lax monoidal adjunctions, two-variable fibrations and the calculus of mates". In: *Proceedings of the London Mathematical Society* 127.4 (2023), pp. 889–957. ISSN: 1460-244X. DOI: 10.1112/plms.12548. (Visited on 10/01/2025) (cit. on p. 235).

[Hau16]   Rune Haugseng. "Bimodules and natural transformations for enriched ∞-categories". In: *Homology, Homotopy and Applications* 18.1 (2016), pp. 71–98. ISSN: 15320073, 15320081. DOI: 10.4310/HHA.2016.v18.n1.a5 (cit. on p. 264).

[Hau18]   Rune Haugseng. "Iterated spans and classical topological field theories". In: *Mathematische Zeitschrift* 289.3 (Aug. 2018), pp. 1427–1488. ISSN: 0025-5874, 1432-1823. DOI: 10.1007/s00209-017-2005-x (cit. on p. 263).

[Hir09]   Philip Hirschhorn. *Model Categories and Their Localizations*. Vol. 99. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, Aug. 24, 2009. ISBN: 978-0-8218-4917-0 978-1-4704-1326-2. DOI: 10.1090/surv/099 (cit. on p. 17).

[HL13]   Michael Hopkins and Jacob Lurie. "Ambidexterity in K(n)-Local Stable Homotopy Theory". In: (2013) (cit. on p. 263).

[HM24]   Elies Harington and Samuel Mimram. "Polynomials in homotopy type theory as a Kleisli category". In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 4 - Proceedings of MFPS XL, 11 (Dec. 2024). ISSN: 2969-2431. DOI: 10.46298/entics.14786. URL: https://entics.episciences.org/14786 (cit. on p. 7).

[HM25]   Eliès Harington and Samuel Mimram. "∞-Categorical Models of Linear Logic". In: *10th International Conference on Formal Structures for Computation and Deduction (FSCD 2025)*. Ed. by Maribel Fernández. Vol. 337. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025, 23:1–23:20. ISBN: 978-3-95977-374-4. DOI: 10.4230/LIPIcs.FSCD.2025.23 (cit. on pp. 7, 259).

[Hof13]   Alexander E. Hoffnung. *Spans in 2-Categories: A monoidal tricategory*. Sept. 18, 2013. DOI: 10.48550/arXiv.1112.0560. arXiv: 1112.0560[math]. URL: http://arxiv.org/abs/1112.0560 (visited on 03/21/2023) (cit. on p. 70).

[HS03]   Martin Hyland and Andrea Schalk. "Glueing and orthogonality for models of linear logic". In: *Theoretical Computer Science*. Category Theory and Computer Science 294.1 (Feb. 15, 2003), pp. 183–231. ISSN: 0304-3975. DOI: 10.1016/S0304-3975(01)00241-9 (cit. on pp. 79, 81, 88).

[HS98]   Martin Hofmann and Thomas Streicher. "The groupoid interpretation of type theory". In: *Twenty Five Years of Constructive Type Theory*. Oxford University Press, Oct. 15, 1998. ISBN: 978-0-19-850127-5 978-0-19-191654-0. DOI: 10.1093/oso/9780198501275.003.0008. URL: https://academic.oup.com/book/41963/chapter/355193131 (visited on 05/29/2025) (cit. on p. 99).

[htt]   Zhen Lin (https://mathoverflow.net/users/11640/zhen-lin). *Is the simplicial nerve a localization?* MathOverflow. URL: https://mathoverflow.net/q/491015 (cit. on p. 207).

[Jac94]   Bart Jacobs. "Semantics of Weakening and Contraction". In: *Annals of Pure and Applied Logic* 69.1 (1994), pp. 73–106 (cit. on p. 271).

[Joy02]   A. Joyal. "Quasi-categories and Kan complexes". In: *Journal of Pure and Applied Algebra*. Special Volume celebrating the 70th birthday of Professor Max Kelly 175.1 (Nov. 8, 2002), pp. 207–222. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(02)00135-4 (cit. on p. 198).

[Joy08]   Andre Joyal. *NOTES ON QUASI-CATEGORIES*. 2008 (cit. on p. 21).

[Joy81]   André Joyal. "Une théorie combinatoire des séries formelles". In: *Advances in Mathematics* 42.1 (Oct. 1, 1981), pp. 1–82. ISSN: 0001-8708. DOI: 10.1016/0001-8708(81)90052-9 (cit. on pp. 86, 173).

[JT84]   André Joyal and Myles Tierney. *An extension of the Galois theory of Grothendieck*. Online-Ausg. Memoirs of the American Mathematical Society volume 309. Providence, R.I: American Mathematical Society, 1984. 1 p. ISBN: 978-0-8218-2312-5 978-1-4704-0722-3 (cit. on pp. 71, 72).

[Kaw73]   Yasuo Kawahara. "Relations in Categories with Pullbacks". In: *Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics* 27.1 (1973), pp. 149–173. DOI: 10.2206/kyushumfs.27.149 (cit. on p. 48).

[Kei78]   William F. Keigher. "Symmetric monoidal closed categories generated by commutative adjoint monads". en. In: *Cahiers de topologie et géométrie différentielle* 19.3 (1978), pp. 269–293 (cit. on p. 269).

[KL21]   Krzysztof Kapulkin and Peter LeFanu Lumsdaine. "The simplicial model of Univalent Foundations (after Voevodsky)". In: *Journal of the European Mathematical Society* 23.6 (Mar. 8, 2021), pp. 2071–2126. ISSN: 1435-9855, 1435-9863. DOI: 10.4171/jems/1050. URL: https://ems.press/doi/10.4171/jems/1050 (cit. on pp. 100, 165).

[KMR25]   Marie Kerjean, Valentin Maestracci, and Morgan Rogers. "Functorial Models of Differential Linear Logic". In: *LIPIcs, Volume 337, FSCD 2025*. Ed. by Maribel Fernández. Vol. 337. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025, 26:1–26:17. ISBN: 978-3-95977-374-4. DOI: 10.4230/LIPICS.FSCD.2025.26 (cit. on p. 272).

[Koc09]   Joachim Kock. "Notes on Polynomial Functors". In: (2009) (cit. on p. 175).

[Kos21]   Roman Kositsyn. *Factorization systems in ∞-categories*. June 8, 2021. DOI: 10.48550/arXiv.2105.14654 (cit. on p. 85).

[Laf88]   Yves Lafont. "Logiques, catégories et machines". PhD thesis. Université Paris 7, 1988 (cit. on p. 39).

[Lai+13]   Jim Laird et al. "Weighted Relational Models of Typed Lambda-Calculi". In: 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science. ISSN: 1043-6871. June 2013, pp. 301–310. DOI: 10.1109/LICS.2013.36. URL: https://ieeexplore.ieee.org/document/6571562 (visited on 05/27/2025) (cit. on pp. 68, 81).

[Lei04]   Tom Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. DOI: 10.1017/CBO9780511525896 (cit. on pp. 69, 226).

[Lem21]   Jean-Simon Pacaud Lemay. "Coderelictions for Free Exponential Modalities". In: *LIPIcs, Volume 211, CALCO 2021* 211 (2021), 19:1–19:21. ISSN: 1868-8969. DOI: 10.4230/LIPICS.CALCO.2021.19 (cit. on pp. 89, 187).

[Lopa]   Adele Lopez. *Double gluing*. URL: https://ncatlab.org/nlab/show/double+gluing (visited on 09/02/2025) (cit. on p. 88).

[Lopb]   Adele Lopez. *Visual Linear Logic*. URL: http://adelelopez.com/visual-linear-logic (visited on 01/08/2023) (cit. on p. 34).

[Lur09]   Jacob Lurie. *Higher topos theory*. Annals of mathematics studies no. 170. OCLC: ocn244702012. Princeton, N.J: Princeton University Press, 2009. 925 pp. ISBN: 978-0-691-14048-3 978-0-691-14049-0 (cit. on pp. 6, 7, 21, 140, 188, 204, 205, 208, 211, 215, 216, 221, 223–225, 235, 250, 251, 253–255, 257, 259).

[Lur17]   Jacob Lurie. *Higher Algebra*. Sept. 18, 2017. URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf (visited on 09/07/2023) (cit. on pp. 6, 7, 21, 153, 188, 204, 224, 225, 227, 228, 230–233, 236–239, 241, 242, 244–246, 251–256, 258, 259, 261, 266–269, 273).

[Lur18]   Jacob Lurie. *Kerodon*. https://kerodon.net. 2018 (cit. on pp. 7, 21, 188, 191, 192, 194, 195, 197–202, 204–215, 217–222, 253, 256).

[Mel03]   Paul-André Melliès. "Categorical models of linear logic revisited". Unpublished manuscript. Oct. 2003. URL: https://hal.science/hal-00154229 (visited on 01/08/2025) (cit. on p. 39).

[Mel09]   Paul-André Melliès. "Categorical semantics of linear logic". In: *Panoramas et synthèses* 27 (2009), pp. 15–215 (cit. on pp. 34–37, 39, 40, 48).

[Mel19]   Paul-Andre Melliès. "Template games and differential linear logic". In: 2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). Vancouver, BC, Canada: IEEE, June 2019, pp. 1–13. ISBN: 978-1-7281-3608-0. DOI: 10.1109/LICS.2019.8785830 (cit. on pp. 5, 70, 165, 187).

[MH25]   Nicholas J. Meadows and Simon Henry. "Higher theories and monads". In: *Higher Structures* 9.1 (May 27, 2025), pp. 227–268. DOI: 10.21136/HS.2025.05 (cit. on pp. 245, 246).

[Mil00]   Stefan Milius. "Relations in categories". PhD thesis. June 15, 2000 (cit. on p. 48).

[MT10]   Paul-André Melliès and Nicolas Tabareau. "Resource modalities in tensor logic". In: *Annals of Pure and Applied Logic* 161.5 (2010), pp. 632–653 (cit. on p. 271).

[MTT09]   Paul-André Melliès, Nicolas Tabareau, and Christine Tasson. "An Explicit Formula for the Free Exponential Modality of Linear Logic". In: *Automata, Languages and Programming*. Ed. by Susanne Albers et al. Berlin, Heidelberg: Springer, 2009, pp. 247–260. ISBN: 978-3-642-02930-1. DOI: 10.1007/978-3-642-02930-1_21 (cit. on p. 250).

[Mur15]   Daniel Murfet. "On Sweedler's cofree cocommutative coalgebra". In: *Journal of Pure and Applied Algebra* 219.12 (Dec. 1, 2015), pp. 5289–5304. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2015.05.012. URL: https://www.sciencedirect.com/science/article/pii/S0022404915001279 (visited on 08/30/2025) (cit. on p. 80).

[Mur17]   Daniel Murfet. *Logic and linear algebra: an introduction*. Jan. 3, 2017. DOI: 10.48550/arXiv.1407.2650 (cit. on p. 79).

[nLa25a]   nLab authors. *commutative algebraic theory*. Feb. 2025. URL: https://ncatlab.org/nlab/revision/commutative+algebraic+theory/39 (cit. on p. 269).

[nLa25b]   nLab authors. *Goodwillie calculus*. Feb. 2025. URL: https://ncatlab.org/nlab/show/Goodwillie+calculus#GoodwillieDerivativeOfExponentialModality (cit. on p. 270).

[nLa26]   nLab authors. *principle of equivalence*. https://ncatlab.org/nlab/show/principle+of+equivalence. Revision 119. Jan. 2026 (cit. on p. 10).

[Oli20]   Federico Olimpieri. "Intersection Types and Resource Calculi in the Denotational Semantics of Lambda-Calculus". PhD thesis. Aix-Marseille Université, Nov. 27, 2020. URL: https://theses.hal.science/tel-03123485 (visited on 03/05/2025) (cit. on pp. 5, 78).

[Pai14]   Valeria de Paiva. "Categorical Semantics of Linear Logic for All". In: *Advances in Natural Deduction: A Celebration of Dag Prawitz's Work*. Dordrecht: Springer Netherlands, 2014, pp. 181–192. ISBN: 978-94-007-7548-0 (cit. on p. 39).

[Pav22]   Dmitri Pavlov. *Answer to "When did the Joyal model structure on simplicial sets originate?"* July 29, 2022. URL: https://mathoverflow.net/a/427514/508115 (visited on 09/24/2025) (cit. on p. 198).

[Rie14]   Emily Riehl. *Categorical homotopy theory*. 2014 (cit. on p. 217).

[Rie24]   Emily Riehl. "On the ∞-topos semantics of homotopy type theory". In: *Bulletin of the London Mathematical Society* 56.2 (Feb. 2024), pp. 461–517. ISSN: 0024-6093, 1469-2120. DOI: 10.1112/blms.12997 (cit. on p. 134).

[Rij25]   Egbert Rijke. *Introduction to Homotopy Type Theory*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2025. ISBN: 978-1-108-84416-1. URL: https://www.cambridge.org/core/books/introduction-to-homotopy-type-theory/0DD31EC06C80797A50ACE807251E80B6 (visited on 08/14/2025) (cit. on p. 111).

[RV16]   Emily Riehl and Dominic Verity. "Homotopy coherent adjunctions and the formal theory of monads". In: *Advances in Mathematics* 286 (Jan. 2, 2016), pp. 802–888. ISSN: 0001-8708. DOI: 10.1016/j.aim.2015.09.011 (cit. on p. 218).

[RV22]   Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. 1st ed. Cambridge University Press, Jan. 31, 2022. ISBN: 978-1-108-93688-0 978-1-108-83798-9. DOI: 10.1017/9781108936880. URL: https://www.cambridge.org/core/product/identifier/9781108936880/type/book (visited on 03/15/2023) (cit. on p. 219).

[See89]   R. A. G. Seely. "Linear logic, ∗-autonomous categories and cofree coalgebras". In: *Categories in computer science and logic*. Contemp. Math. American Mathematical Society, 1989, pp. 371–382 (cit. on pp. 5, 40).

[Shu19]   Michael Shulman. *All (∞, 1)-toposes have strict univalent universes*. version: 2. Apr. 26, 2019. arXiv: 1904.07004[math]. (Visited on 02/01/2023) (cit. on pp. 92, 122).

[Sta13]   Michael Stay. "Compact Closed Bicategories". In: *Theory and Applications of Categories* 31 (Jan. 2013). DOI: 10.70930/tac/msroyxdc (cit. on p. 153).

[Str20]   Ross Street. "Polynomials as spans". In: *Cahiers de topologie et géométrie différentielle catégoriques* LXI-2 (2020), pp. 113–153. URL: http://arxiv.org/abs/1903.03890 (cit. on p. 264).

[Str80]   Ross Street. "Fibrations in bicategories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 21.2 (1980). Publisher: Dunod éditeur, publié avec le concours du CNRS, pp. 111–160. ISSN: 1245-530X. URL: https://eudml.org/doc/91227 (visited on 08/13/2025) (cit. on p. 28).

[Str81]   Ross Street. "Conspectus of variable categories". In: *Journal of Pure and Applied Algebra* 21.3 (June 1, 1981), pp. 307–338. ISSN: 0022-4049. DOI: 10.1016/0022-4049(81)90021-9. URL: https://www.sciencedirect.com/science/article/pii/0022404981900219 (visited on 08/13/2025) (cit. on p. 28).

[Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics lecture note series. New York, NY: Benjamin, 1969. 336 pp. ISBN: 978-0-8053-9255-5 978-0-8053-9254-8 (cit. on p. 80).

[Tri06] Todd Trimble. *Notes on tetracategories*. Oct. 2006. URL: `https://math.ucr.edu/home/baez/trimble/tetracategories.html` (visited on 08/08/2025) (cit. on p. 12).

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `https://homotopytypetheory.org/book`, 2013 (cit. on pp. 12, 92, 94–96, 101–104, 106–109, 111, 112, 114–122, 125, 126, 142, 186).

[VZ14] Benoit Valiron and Steve Zdancewic. "Modeling Simply-Typed Lambda Calculi in the Category of Finite Vector Spaces". In: *Scientific Annals of Computer Science* 24.2 (2014), pp. 325–368. ISSN: 22482695. DOI: `10.7561/SACS.2014.2.325` (cit. on pp. 79, 81).

[Whi49] J H C Whitehead. "COMBINATORIAL HOMOTOPY. I". In: (1949) (cit. on p. 15).

[Win99] Glynn Winskel. "A Linear Metalanguage for Concurrency". In: *Algebraic Methodology and Software Technology*. Ed. by Armando M. Haeberer. Berlin, Heidelberg: Springer, 1999, pp. 42–58. ISBN: 978-3-540-49253-5. DOI: `10.1007/3-540-49253-4_6` (cit. on p. 48).

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